

THE CAPACITY OF ELEMENTS OF BANACH ALGEBRAS

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PREFACE

As its name suggests, this thesis is an account of the recent theory of the capacity of elements of Banach algebras. The first chapter contains a summary of the background theory, other than fundamentals, used, and consists mainly of perturbation theory of linear operators and certain properties of strictly singular operators. This chapter relies heavily on the work of T. Kato, both in his own papers and the book by S. Goldberg "Unbounded Linear Operators".

Chapter 2 introduces the notion of capacity, following Halmos in his paper "Capacity in Banach algebras", and several small new results are proved, and counterexamples given, to tidy up "loose ends". The question of the capacity of the sum of two quasia algebraic elements (i.e. ones with capacity zero) is raised, and a partial solution given. The perturbation theory of Chapter 1 is applied to show the equality of the capacity of the spectrum and the Fredholm spectrum of an operator on a Banach space, whence it is shown that if J is a closed two-sided ideal of $B(X)$ containing only Riesz operators, then perturbation by an element of J leaves the capacity invariant; this is true, in particular, for compact operators. A converse theorem is proved for Hilbert space.

Chapter 3 introduces the new concept of the joint capacity of an r -tuple of elements of a commutative Banach algebra, and develops the theory of this notion. Much of the theory parallels, in a weaker form, that of the original concept, but there are significant differences. Finally, a perturbation theorem, similar to the original one is proved for the joint capacity.

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CONTENTS

Notation	page	iv
Chapter 1. LINEAR TRANSFORMATIONS OF BANACH SPACES		
§1. Projections and subspaces		1
§2. Compact operators and related topics		4
§3. Perturbation theory		8
§4. Strictly singular operators		26
Chapter 2. THE CAPACITY OF AN ELEMENT OF A BANACH ALGEBRA		
§1. Definitions and fundamentals		33
§2. Spectral capacity		36
§3. Quasialgebraic elements and operators		41
§4. Perturbations		51
Chapter 3. JOINT CAPACITY		
§1. Introduction		58
§2. Basic definitions and properties		60
§3. Spectral capacity		71
§4. A perturbation theorem		79
References		85

NOTATION

The following notation will be used as standard, without further explanation:-

\mathbb{R}, \mathbb{C} the real and complex numbers respectively.

$E \subseteq F$ E is a subset of F .

$E \subset F$ E is a subset of F , and $E \neq F$.

$x \in E$ x is an element of E .

$\mathcal{C}E$ the complement of E .

$E \setminus F$ the set $E \cap \mathcal{C}F$.

Where X and Y are normed linear spaces:-

$B(X, Y)$ the set of bounded linear maps from X to Y .

$B(X)$ the set of bounded linear maps from X to itself.

$K(X)$ (where X is a Banach space) the set of compact operators in $B(X)$.

If A is a Banach algebra, and $a \in A$, we write:-

$\text{Sp } a$ the spectrum of a .

$\rho(a)$ the spectral radius of a .

We shall use $\text{Sp } T$, $\rho(T)$ for the spectrum and spectral radius, respectively, of a bounded operator T .

CHAPTER 1

LINEAR TRANSFORMATIONS OF BANACH SPACES

§ 1. Projections and subspaces.

If E is a closed linear subspace of a Banach space X , we say E is complemented if it has a closed complement, that is, if there exists a closed subspace F of X such that

$$X = E \oplus F.$$

It is a theorem of linear algebra that every subspace has a complement, but not necessarily a closed one; we do not give an example of a closed subspace of a Banach space which is not complemented.

In this chapter all linear spaces X, Y will, unless otherwise stated, be Banach spaces.

Theorem 1.1

A closed subspace E of a Banach space X is complemented if and only if there exists a bounded projection with range E .

Proof:

Let $P \in B(X)$ be a projection, with range E . Denote by F the subspace $(I - P)X$, which is closed since P is bounded. Now

$$X = E \oplus F,$$

so E is complemented.

Conversely, assume that $X = E \oplus F$ where E and F are closed subspaces. For any element $x \in X$ we may write

$$x = x_1 + x_2,$$

where $x_1 \in E$ and $x_2 \in F$, the resolution being unique. We now define P by the equation

$$Px = x_1 .$$

It is clear that P is well defined, linear and idempotent; we must show that it is bounded.

Let G be the graph of P ,

$$G = \{ (x, Px) : x \in X \} .$$

Let $(x^{(n)}, Px^{(n)}) \in G$ for $n = 1, 2, \dots$, and let

$$(x^{(n)}, Px^{(n)}) \rightarrow (x, y) .$$

Then $x^{(n)} \rightarrow x$, $Px^{(n)} \rightarrow y$ as $n \rightarrow \infty$, and so $y \in E$ since E is closed and $Px^{(n)} \in E$ ($n = 1, 2, \dots$). Moreover,

$$x^{(n)} - Px^{(n)} \rightarrow x - y \in F ,$$

since F is closed. Now $y \in E$ and $x - y \in F$, so $Px = y$, and $(x, y) \in G$, that is G is closed. The closed graph theorem now shows that P is bounded, and our theorem is proved.

By a suitable choice of functionals and basis elements, it is easy to show that any finite - dimensional subspace is complemented by constructing a projection with the subspace as its range.

The following theorem of Kato [8] will be used frequently.

Theorem 1.2

If $T \in B(X, Y)$ and TX is of finite codimension in Y , then TX is closed.

Proof:

Since TX has finite codimension, we can write

$$Y = TX \oplus M ,$$

where M is a finite - dimensional subspace of Y . We form the (exterior) direct sum

$$\tilde{X} = X \oplus M,$$

with the natural algebraic operations and the norm

$$\| (x, m) \| = \| x \| + \| m \|,$$

where $x \in X$ and $m \in M$. With this norm, \tilde{X} is a Banach space.

Define $T \in B(\tilde{X}, Y)$ by

$$\tilde{T}(x, m) = Tx + m.$$

Clearly, \tilde{T} maps \tilde{X} onto Y .

Assume that T is one-to-one; then \tilde{T} is one-to-one and onto, so Banach's isomorphism theorem shows that \tilde{T} has a bounded inverse.

In particular, there exists a constant $\kappa > 0$ such that

$$\| \tilde{T}(x, m) \| \geq \kappa \| (x, m) \| \quad \forall (x, m) \in \tilde{X}.$$

Putting $m = 0$, we obtain

$$\| Tx \| = \| \tilde{T}(x, 0) \| \geq \kappa \| (x, 0) \| = \kappa \| x \| \quad \forall x \in X,$$

whence TX is closed.

Suppose, on the other hand, that T is not one-to-one. Let $Z = X/T^{-1}(0)$ and write T_0 for the induced map $T_0: Z \rightarrow Y$. Since $T^{-1}(0)$ is closed, Z is a Banach space and T_0 is bounded, one-to-one and has range TX , which is of finite codimension in Y . Applying the previous argument to T_0 now shows that TX is closed.

To conclude this section we give a useful lemma.

Lemma 1.3

Let E be a closed subspace and F a finite-dimensional subspace of a Banach space X . Then $E \oplus F$ is closed.

Proof:

Let $\pi: X \rightarrow X/E$ be the natural epimorphism. $\pi(F)$ is finite-dimensional and hence closed in X/E , so $\pi^{-1}(\pi(F)) = E \oplus F$ is closed since π is continuous.

§ 2. Compact operators and related topics

We shall assume the results of the standard Riesz-Schauder theory for compact operators on a Banach space [12], that is, the following results:

- If $K \in K(X)$, then
- (i) $\text{Sp } K$ is at most countable, and 0 is its only accumulation point, if any exist.
 - (ii) If $\lambda \in \text{Sp } K \setminus \{0\}$, then λ is an eigenvalue and the space $N^{(\lambda)} = \{x: Kx = \lambda x\}$ has finite dimension.
 - (iii) If $\lambda \in \text{Sp } K \setminus \{0\}$, then $(K - \lambda I)X$ is closed and of finite codimension in X .
 - (iv) Let $N_k^{(\lambda)} = \{x: (K - \lambda I)^k x = 0\}$ and $R_k^{(\lambda)} = (K - \lambda I)^k X$, where k is a non-negative integer. Then there exists an integer n such that

$$\begin{aligned} \{0\} &= N_0^{(\lambda)} \subset N_1^{(\lambda)} \subset \dots \subset N_n^{(\lambda)} = N_{n+1}^{(\lambda)} = \dots, \\ X &= R_0^{(\lambda)} \supset R_1^{(\lambda)} \supset \dots \supset R_n^{(\lambda)} = R_{n+1}^{(\lambda)} = \dots, \end{aligned}$$

and

$$X = N_n^{(\lambda)} \oplus R_n^{(\lambda)}.$$

The properties listed above are neither mutually independent nor minimal; for example, (i) follows from the others.

Definition.

An operator $T \in B(X)$ satisfying (i) to (iv) above is said to be a Riesz operator. It is not difficult to see that the following conditions are equivalent to (i) to (iv), but more useful in determining whether or not an operator $T \in B(X)$ is Riesz:

- (i) If $\lambda \in \text{Sp } T \setminus \{0\}$, then λ is an eigenvalue of T .
- (ii) $\forall \lambda \neq 0$, there exists a finite rank projection $P \in B(X)$ such that $PT = TP$, $(T - \lambda I)P$ is nilpotent and the restriction of $T - \lambda I$ to the subspace $(I - P)X$ is one-to-one and maps onto $(I - P)X$.

The range of P is the subspace $N_n^{(\lambda)}$ of the previous page.

Riesz operators have been characterised by Ruston [14] in the following way: X is a Banach space.

Theorem 1.4 (Ruston characterisation).

Let $T \in B(X)$, and denote by \tilde{T} the operator induced by T in $B(X)/K(X)$. Then T is a Riesz operator if and only if \tilde{T} is quasinilpotent.

At this stage it is convenient to introduce the set of Fredholm mappings and the Fredholm spectrum.

Definitions.

$T \in B(X, Y)$ is said to be Fredholm iff

- (i) $\{x: Tx = 0\}$ has finite dimension, and
- (ii) TX is of finite codimension in Y .

We shall denote the set of Fredholm mappings in $B(X, Y)$ by $\Phi(X, Y)$; if X and Y coincide, we shall write $\Phi(X)$ instead of $\Phi(X, X)$.

Notice that, by Theorem 1.2, every Fredholm mapping has closed range.

If $T \in B(X)$, we define the Fredholm spectrum of T , which we denote by $FSp T$, as

$$FSp T = \{ \lambda \in \mathbb{C} : T - \lambda I \notin \Phi(X) \} .$$

Lemma 1.5

If $\tau: B(X) \rightarrow B(X)/K(X)$ is the canonical epimorphism, then

$$Sp \tau(T) = FSp T .$$

In particular, $FSp T$ is closed.

Proof:

Since τ is an identity-preserving epimorphism, we need only show that $0 \in Sp \tau(T)$ if and only if $0 \in FSp T$.

Assume $0 \notin Sp \tau(T)$; then $\exists S \in B(X)$ such that

$$\tau(S)\tau(T) = \tau(T)\tau(S) = \tau(I) .$$

Therefore,

$$ST = I + K_1 ,$$

$$TS = I + K_2 , \quad (K_1, K_2 \in K(X)) ,$$

whence we have

$$\{x: Tx = 0\} \subseteq \{x: STx = 0\} = \{x: (I + K_1)x = 0\} ,$$

$$(I + K_2)X = TSX \subseteq TX ,$$

so, by the Riesz-Schauder theory, the nullspace of T has finite dimension and its range finite codimension, that is $T \in \Phi(X)$ and $0 \notin FSp T$.

Conversely, assume $0 \notin FSp T$, so T has finite-dimensional nullspace and range of finite codimension. Let P be a bounded projection onto $T^{-1}(0)$ and $I - Q$ onto TX . Both P and Q

have finite rank and are therefore compact.

Let $T_0: (I - P)X \rightarrow TX$ be the restriction of T . T_0 is one-to-one and maps onto the Banach space TX , since T has closed range, so Banach's isomorphism theorem shows the existence of $S_0 \in B(TX, (I - P)X)$ such that

$$S_0 T x = x \quad \forall x \in (I - P)X .$$

Let $S = S_0(I - Q)$, so $S \in B(X)$ and we have in turn

$$ST(I - P) = I - P ,$$

$$\tau(S)\tau(T) = \tau(I) ,$$

since $\tau(P) = 0$. Moreover,

$$TS_0 x = x \quad \forall x \in TX = (I - Q)X ,$$

whence

$$TS = I - Q ,$$

$$\tau(T)\tau(S) = \tau(I) ,$$

so $\tau(T)$ is invertible in $B(X)/K(X)$ and $0 \notin \text{Sp } \tau(T)$.

The final remark in the statement follows since $B(X)/K(X)$ is a Banach algebra, so $\tau(T)$ has closed spectrum.

From the last Lemma we may restate the Ruston characterisation.

Theorem 1.6

$T \in B(X)$ is a Riesz operator if and only if $\text{FSp } T = \{0\}$.

Theorem 1.7

Let J be any closed two-sided ideal of $B(X)$ such that J contains only Riesz operators. Then if $\pi: B(X) \rightarrow B(X)/J$ is the canonical epimorphism, we have

$$\text{Sp } \pi(T) = \text{FSp } T .$$

Proof:

Since π is an identity-preserving epimorphism, we need only show that $0 \in \text{Sp } \pi(T)$ if and only if $0 \in \text{FSp } T$.

Assume that $0 \notin \text{Sp } \pi(T)$. Then $\exists S \in B(X)$ such that

$$ST = I + R_1,$$

$$TS = I + R_2, \quad (R_1, R_2 \in J).$$

But J contains only Riesz operators, so R_1 and R_2 are Riesz, and the same argument as in Lemma 1.5 shows that T is Fredholm, that is, that $0 \notin \text{FSp } T$.

Now suppose that $0 \notin \text{FSp } T$. It follows that $T^{-1}(0)$ has finite dimension and TX finite codimension. Let P and Q be bounded projections, and let $T^{-1}(0)$ and TX be the ranges of P and $I - Q$ respectively. Then P and Q are finite-rank operators and so belong to J , since any two-sided ideal of $B(X)$ contains the operators of finite rank. We now proceed in the same way as in Lemma 1.5, and define $T_0: (I - P)X \rightarrow TX$, the restriction of T to $(I - P)X$, and we see that $\pi(T)$ is invertible in $B(X)/J$ since P and Q both lie in J . Hence $0 \notin \text{Sp } \pi(T)$.

§ 3. Perturbation theory

The results in this section are given strictly for use later in the thesis, and are not given in their full generality, since most of the theorems apply also to closed operators, which will not concern us. For a full treatment see [3], [4] and [8]. We shall start with some definitions.

Definitions

If $T \in B(X, Y)$, where X and Y are normed linear spaces, define the nullity of T , written $\text{nul } T$, to be the dimension of $T^{-1}(0)$, and the deficiency of T , $\text{def } T$, to be the codimension of TX , where these are allowed to take the value ∞ if the appropriate dimension is infinite. Where at least one of $\text{nul } T$, $\text{def } T$ takes a finite value, we define the index of T , written $\kappa(T)$ as

$$\kappa(T) = \text{nul } T - \text{def } T,$$

with the conventions that $\infty - n = \infty$, $n - \infty = -\infty$, for any integer n .

If $T \in B(X, Y)$ we shall denote by T^* the adjoint operator $T^*: Y^* \rightarrow X^*$ defined by

$$(T^*f)(x) = f(Tx) \quad \forall x \in X, f \in Y^*,$$

where X^* denotes the dual of X .

The map $T \rightarrow T^*$ is a linear isometry.

Lemma 1.8 (Banach)

If X is a Banach space and Y a normed linear space with $T \in B(X, Y)$, then

$$\{y \in Y: \|y\| < r\} \subseteq \overline{\{Tx: \|x\| \leq 1\}}$$

implies that

$$\{y \in Y: \|y\| < r\} \subseteq \{Tx: \|x\| \leq 1\}.$$

Proof:

Denote by $B_X(r)$ the set $\{x \in X: \|x\| < r\}$, and by $\overline{B_X(r)}$ the set $\{x \in X: \|x\| \leq r\}$.

Let $\|y\| < r$, say $\|y\| < (1 - \epsilon)r$, where $0 < \epsilon < 1$.

Let $\tilde{y} = (1 - \epsilon)^{-1}y$ so that \tilde{y} is a member of $\overline{B_X(r)}$.

There exists $x_0 \in X$ such that $\|x_0\| \leq 1$ and $\|\tilde{y} - Tx_0\| < r\epsilon$.

Now assume we have found $x_0, \dots, x_n \in X$ with $\|x_k\| \leq \epsilon^k$, for $k = 0, 1, \dots, n$ and $\|\tilde{y} - Tx_0 - Tx_1 - \dots - Tx_n\| < r\epsilon^{n+1}$.

Since

$$\epsilon^{-(n+1)}(\tilde{y} - Tx_0 - \dots - Tx_n) \in B_Y(r) \subseteq \overline{\{Tx: \|x\| \leq 1\}},$$

$\exists \tilde{x}_{n+1} \in X$ with $\|\tilde{x}_{n+1}\| \leq 1$ and

$$\|\epsilon^{-(n+1)}(\tilde{y} - Tx_0 - \dots - Tx_n) - \tilde{x}_{n+1}\| < r\epsilon,$$

so, putting $x_{n+1} = \tilde{x}_{n+1}\epsilon^{n+1}$, we have $\|x_{n+1}\| \leq \epsilon^{n+1}$ and

$$\|\tilde{y} - Tx_0 - \dots - Tx_{n+1}\| < r\epsilon^{n+2},$$

and, by induction, we obtain a sequence of such elements x_n . Put

$s_n = x_0 + \dots + x_n$. Then $\|s_n\| \leq 1 + \epsilon + \dots + \epsilon^n = (1 - \epsilon^{n+1})/(1 - \epsilon)$

so $\{s_n\}$ converges, since it is absolutely convergent and X is a

Banach space. Let $s_n \rightarrow s$ as $n \rightarrow \infty$; clearly $\|s\| \leq 1/(1 - \epsilon)$.

Now,

$$\|\tilde{y} - Ts\| = \lim_{n \rightarrow \infty} \|\tilde{y} - Ts_n\| = 0,$$

so $\tilde{y} = Ts$, so $y = (1 - \epsilon)\tilde{y} = T\{(1 - \epsilon)s\}$ and $\|(1 - \epsilon)s\| \leq 1$.

Thus y , an arbitrary element of $B_Y(r)$ belongs to the set $\{Tx: \|x\| \leq 1\}$, and so

$$\{y \in Y: \|y\| < r\} \subseteq \{Tx: \|x\| \leq 1\},$$

which is the required result.

Definition

If $T \in B(X, Y)$ we say T has a bounded left inverse if there exists a linear map $S \in B(Y, X)$ such that $STx = x \quad \forall x \in X$. We do not demand that S be everywhere defined on Y .

Lemma 1.9

If X and Y are Banach spaces, $T \in B(X, Y)$ and T^* is one-

to-one with a bounded left inverse, then T maps X onto Y .

Proof:

To show that T maps onto Y , we need only show that, for some positive r , $T(B_X(1)) \supseteq B_Y(r)$. By the previous lemma, it is enough to prove that

$$\overline{T(B_X(1))} \supseteq B_Y(r) . \quad (1)$$

We shall denote by T^0 the left inverse of T^* , and show that (1) holds with $r = 1/\|T^0\|$.

Suppose that this is not the case; then there exists $y_0 \in Y$ with $\|y_0\| < 1/\|T^0\|$ such that $y_0 \notin \overline{T(B_X(1))}$. Using a hyperplane separation theorem [2, p417] since $\overline{T(B_X(1))}$ is closed and convex, we see that there exists $f \in Y^*$ and $\epsilon > 0$ with

$$\begin{aligned} \operatorname{Re} f(y_0) &= c > c - \epsilon \geq \sup \{ \operatorname{Re} f(y) : y \in \overline{T(B_X(1))} \} \\ &= \sup \{ \operatorname{Re} f(Tx) : \|x\| \leq 1 \} \\ &= \sup \{ |f(Tx)| : \|x\| \leq 1 \} \\ &= \|T^* f\| . \end{aligned}$$

Therefore

$$\begin{aligned} \|T^* f\| &< \operatorname{Re} f(y_0) \\ &\leq |f(y_0)| \\ &\leq \|f\| \cdot \|y_0\| \\ &\leq \|T^0\| \cdot \|T^* f\| \cdot \|y_0\| , \end{aligned}$$

since $f = T^0 T^* f$. But T^* is assumed to be one-to-one, so $T^* f \neq 0$ and we obtain

$$1 < \|T^0\| \cdot \|y_0\| \leq 1 ,$$

a contradiction. Thus our assumption that $y_0 \notin \overline{T(B_X(1))}$ is untenable and the Lemma is proved.

Theorem 1.10

If X and Y are Banach spaces and $T \in B(X, Y)$ then the range of T is closed if and only if the range of T^* is closed.

Proof:

Define a map $T_1 \in B(X, Z)$ where $Z = \overline{TX}$ by the equation

$$T_1 x = Tx \quad \forall x \in X.$$

We shall show that $T_1^{**} Z^* = T^* Y^*$.

$$\begin{aligned} x^* \in T^* Y^* &\Leftrightarrow \exists y^* \in Y^* \text{ such that } T^* y^* = x^* \\ &\Leftrightarrow \exists y^* \in Y^* \text{ such that } y^*(Tx) = x^*(x) \quad \forall x \in X \\ &\Rightarrow \exists z^* \in Z^* \text{ such that } z^*(T_1 x) = x^*(x) \quad \forall x \in X \\ &\Rightarrow x^* \in T_1^* Z^*. \end{aligned}$$

(A suitable z^* would be $y^*|_Z$, which shows its existence.)

Now let $x^* = T_1^* z^*$. To show the equality of $T^* Y^*$ and $T_1^* Z^*$, it will be enough to show that $x^* \in T^* Y^*$. By the Hahn-Banach theorem, there exists $y^* \in Y^*$ such that $\|y^*\| = \|z^*\|$ and $y^*|_Z = z^*$.

Then we have

$$\begin{aligned} x^*(x) &= z^*(T_1 x) = y^*(Tx) \quad \forall x \in X \\ &= (T^* y^*)(x) \quad \forall x \in X, \end{aligned}$$

so $x^* \in T^* Y^*$, and we have

$$T^* Y^* = T_1^* Z^*.$$

Since T and T_1 have ranges which are isometrically isomorphic, the theorem will be proved if we can show that T_1 has closed range if and only if T_1^* has closed range.

Assume that TX and hence $T_1 X$ is closed, so that T_1 is surjective and T_1^* is one-to-one; for if $T_1^* f = 0$ then it follows that $f(TX) = \{0\}$, whence $f = 0$.

We shall show that T_1^* has a bounded left inverse. Suppose the contrary, that is, suppose there exists a sequence $\{z_n^*\}$ of elements of Z^* with $\|z_n^*\| = 1$ ($n = 1, 2, \dots$) and for which $\|T_1^* z_n^*\| \rightarrow 0$ as $n \rightarrow \infty$. Write

$$u_n^* = \frac{1}{\sqrt{\|T_1^* z_n^*\|}} z_n^*, \quad (n = 1, 2, \dots),$$

so that

$$\|u_n^*\| \rightarrow \infty, \quad \|T_1^* u_n^*\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now, if $x \in X$,

$$u_n^*(T_1 x) = (T_1^* u_n^*)(x) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

whence, since T_1 is surjective,

$$u_n^*(z) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \forall z \in Z.$$

Therefore

$$\sup_n \|u_n^*(z)\| < \infty \quad \forall z \in Z,$$

and the uniform boundedness theorem shows that $\sup_n \|u_n^*\| < \infty$, a contradiction. Thus T_1^* has a bounded left inverse and hence closed range, since $\exists \kappa > 0$ with $\|T_1^* z^*\| \geq \kappa \|z^*\|$.

Conversely, let $T^* Y$ be closed; we pass to T_1 as before and notice that $T_1^* Z^*$ is closed. Since $T_1 X$ is dense in Z by the definition of Z , T_1^* is one-to-one. It follows, since T_1^* is one-to-one with closed range, that T_1 has a bounded left inverse, and Lemma 1.9 shows that T_1 is surjective, that is, that T has closed range.

Corollary

If T has closed range, then

$$\text{nul } T = \text{def } T^*, \quad \text{def } T = \text{nul } T^*.$$

Proof:

Let $\text{mul } T = \alpha$. Then there exists a set $\{x_k\}_{k=1}^{\alpha}$ of linearly independent elements of X for which $Tx_k = 0$ ($k = 1, 2, \dots, \alpha$). To save a special case, we adopt the convention here that if $\alpha = \infty$, x_1, \dots, x_{α} will denote a sequence x_1, x_2, \dots and proceed in the same way.

Choose $f_k \in X^*$ such that

$$f_k(x_k) \neq 0 \quad (k = 1, \dots, \alpha),$$

$$f_k(x_i) = 0 \quad (i < k).$$

The set $\{f_k\}_{k=1}^{\alpha}$ is linearly independent. We show that f_k does not belong to T^*Y^* ($k = 1, \dots, \alpha$). Assume the contrary, that is, that there exists $g_k \in Y^*$ such that $f_k = T^*g_k$ for some k . Then

$$0 \neq f_k(x_k) = (T^*g_k)(x_k) = g_k(Tx_k) = 0.$$

This contradiction shows that $f_k \notin T^*Y^*$ ($k = 1, 2, \dots, \alpha$) so, that

$$\text{def } T^* \geq \alpha.$$

Now let $\text{def } T^* = \beta$, and use the same conventions if $\beta = \infty$. As in previous cases, we define $T_1: X \rightarrow Z = TX$; T_1 is surjective and $T_1Z^* = T^*Y^*$. Let $\tilde{T}: X/T^{-1}(0) \rightarrow Z$ be the induced mapping, and let $\pi: X \rightarrow X/T^{-1}(0)$ be the natural epimorphism. \tilde{T} is one-to-one and surjective, so by Banach's isomorphism theorem it has a bounded inverse, \tilde{T}^{-1} . This implies that \tilde{T}^* is surjective, for let $g \in \{X/T^{-1}(0)\}^*$ and define $z^* \in Z^*$ by

$$z^*(z) = g(\tilde{T}^{-1}z) \quad \forall z \in Z.$$

Then

$$z^*(\tilde{T}\pi x) = g(\tilde{T}^{-1}\tilde{T}\pi x) = g(\pi x) \quad \forall x \in X,$$

since \tilde{T} is surjective. Therefore

$$\hat{T}^*z^* = g,$$

and so \hat{T}^* is surjective, that is

$$\hat{T}^* Z^* = \{X/T^{-1}(0)\}^*.$$

Further, $\tilde{T}\pi = T_1$, so that $T_1^* = \pi^* \tilde{T}^*$, where $*$ denotes the dual in the appropriate space. It follows that

$$\begin{aligned} T Y^* &= T_1 Z^* && \text{(as before)} \\ &= \pi^* \tilde{T}^* Z^* \\ &= \pi^* \{X/T^{-1}(0)\}^*. \end{aligned} \quad (1)$$

Let $g \in \{X/T^{-1}(0)\}^*$; then

$$(\pi^* g)(x) = g(\pi x) = 0 \quad \forall x \in T^{-1}(0),$$

so

$$\pi^* \{X/T^{-1}(0)\}^* \subseteq \{f \in X^* : f(T^{-1}(0)) = \{0\}\}. \quad (2)$$

To prove the opposite inclusion, let $f \in X^*$ with $f(T^{-1}(0)) = \{0\}$ and define

$$g(\pi x) = f(x) \quad \forall x \in X.$$

g is a well-defined member of $\{X/T^{-1}(0)\}^*$ and $\pi^* g = f$, whence the converse inclusion to (2) holds, and we have

$$\pi^* \{X/T^{-1}(0)\}^* = \{f \in X^* : f(T^{-1}(0)) = \{0\}\}.$$

Taking this with (1) we obtain

$$T Y^* = \{f \in X^* : f(T^{-1}(0)) = \{0\}\}. \quad (3)$$

Now let $\{f_1, \dots, f_n\}$ be any set of elements of X^* which are linearly independent modulo $T Y^*$, that is, no non-zero linear combination of f_1, \dots, f_n lies in $T Y^*$. We can find such a set if and only if $n \leq \dim T Y^*$. Let f_k^0 denote the restriction of f_k to $T^{-1}(0)$. Since no linear combination of f_1, \dots, f_n lies in $T Y^*$ no such linear combination (other than zero) annihilates $T^{-1}(0)$ by (3), and hence $\{f_1^0, \dots, f_n^0\}$ is a linearly independent set of elements of the dual of $T^{-1}(0)$, so $\dim T^{-1}(0) \geq n$.

If $\text{def } T^* < \infty$, put $n = \text{def } T^*$, so $\text{nul } T \geq \text{def } T^*$. If $\text{def } T^*$ has the value ∞ , then for all positive integers n , $n \leq \text{def } T^*$, so by the above we obtain $\text{nul } T \geq n$, hence $\text{nul } T = \infty$. In either case $\text{nul } T \geq \text{def } T^*$, and, combining this with the reverse inequality already obtained, we have

$$\text{nul } T = \text{def } T^*.$$

The equation $\text{nul } T^* = \text{def } T$ is proved similarly using the identity

$$T^{*-1}(0) = \{y^* \in Y^* : y^*(TX) = \{0\}\}.$$

Definition

We define the lower bound $\gamma(T)$ of $T \in B(X, Y)$ as

$$\gamma(T) = \inf \left\{ \frac{\|Tx\|}{d(x, T^{-1}(0))} : x \in X \right\},$$

where we adopt the convention that $0/0 = \infty$, and

$$d(x, T^{-1}(0)) = \inf \{ \|x - y\| : y \in T^{-1}(0) \}.$$

If T is one-to-one, it is clear that $\gamma(T) > 0$ if and only if T has a bounded left inverse, that is iff T has closed range. If T is not one-to-one, let $\tilde{T}: X/T^{-1}(0) \rightarrow Y$ be the induced map on $X/T^{-1}(0)$. \tilde{T} is one-to-one, and since $d(x, T^{-1}(0)) = \|x + T^{-1}(0)\|$, $\gamma(\tilde{T}) = \gamma(T)$, whence \tilde{T} , and so also T , has closed range if and only if $\gamma(T) > 0$.

Lemma 1.11

Let M, N be subspaces of X with $\dim M > \dim N$. Then there exists a non-zero $x \in M$ for which $\|x\| = d(x, N)$.

Proof:

Since $\dim N < \infty$, we need only prove the lemma for $\dim N = n$, $\dim M = n + 1$, where n is a positive integer. We shall assume first that the unit ball of $M + N$ is strictly convex, i.e. if x and y are linearly independent that $\|x + y\| < \|x\| + \|y\|$.

Since N has finite dimension, for each $x \in M$, there is a $z \in N$ such that $\|x - z\| = d(x, N)$. This z is unique since if

$\|x - z_1\| = \|x - z_2\| = d(x, N)$ with z_1, z_2 in N , we have

$$\begin{aligned} d(x, N) &\leq \|x - \frac{1}{2}(z_1 + z_2)\| = \frac{1}{2}\|(x - z_1) + (x - z_2)\| \\ &\leq \frac{1}{2}d(x, N) + \frac{1}{2}d(x, N), \end{aligned}$$

with equality only if $x - z_1$ and $x - z_2$ are linearly dependent,

by the strict convexity. Define $\phi(x)$ to be this unique point z .

Clearly $\phi: M \rightarrow N$ and $\phi(-x) = -\phi(x)$. Let $x_n \rightarrow x$; $\{\phi(x_n)\}$,

being bounded (for $\|\phi(x)\| \leq 2\|x\|$), has a convergent subsequence,

say $\phi(x_{n_k}) \rightarrow y$. Then

$$\|x - y\| = \lim \|x_{n_k} - \phi(x_{n_k})\| = \lim d(x_{n_k}, N) = d(x, N),$$

so $\phi(x_{n_k}) \rightarrow \phi(x)$. ϕ is thus continuous, for if not, there is $\epsilon > 0$

such that $\|\phi(x) - \phi(y_n)\| \geq \epsilon$ for some y_n with $\|y_n - x\| < \frac{1}{n}$,

and this is inconsistent with the above.

We have to show that, for some non-zero x in M , $\phi(x) = 0$.

Let e_1, \dots, e_{n+1} span M and f_1, \dots, f_n span N , and let Σ be

the unit sphere of M . Write $\Gamma: S_{2n+1} \rightarrow \Sigma$ and $\Delta: N \rightarrow \mathbb{R}^{2n+1}$

(where S_{2n+1} is the unit sphere in \mathbb{R}^{2n+2}) where

$$\Gamma(\alpha_1, \beta_1, \dots, \alpha_{n+1}, \beta_{n+1}) = \left\{ \sum_{j=1}^{n+1} (\alpha_j + i\beta_j) e_j \right\} / \left\| \sum_{j=1}^{n+1} (\alpha_j + i\beta_j) e_j \right\|$$

$$\Delta\left(\sum_{j=1}^n (\alpha_j + i\beta_j) f_j\right) = (\alpha_1, \beta_1, \dots, \alpha_n, \beta_n, 0).$$

Γ is continuous, by the compactness of Σ , and Δ is also

continuous, so if $\theta = \Delta \circ \phi \circ \Gamma$, θ is continuous and has the property

$$\theta(-w) = -\theta(w) \quad \forall w \in S_{2n+1}.$$

But $\theta: S_{2n+1} \rightarrow \mathbb{R}^{2n+1}$, so by the Borsuk-Ulam Theorem [18],

$\theta(w) = \theta(-w)$ for some w , that is, $\theta(w) \approx 0$. But then $\phi\{I(w)\} = 0$ and $I(w) \in M$, so the result is proved in the strictly convex case.

Now remove the assumption of strict convexity. Let $\epsilon > 0$ and let $|\cdot|_1$ be any strictly convex norm in $M + N$; since $M + N$ has finite dimension, there exists $k < \infty$ for which $|x|_1 \leq k\|x\|$ for all x in $M + N$. Now let $|x| = \|x\| + \delta|x|_1$ for all x , where $k\delta < \epsilon$. Then $|x + y| < |x| + |y|$ if x and y are linearly independent (since this is true of $|\cdot|_1$) and we have $\|x\| \leq |x| \leq (1 + \epsilon)\|x\|$, where $|\cdot|$ is strictly convex. Therefore there exists, by the first part of the proof, a y in M such that $|y| = \inf\{|y - x| : x \in N\}$, hence

$$\|y\| \leq |y| \leq (1 + \epsilon) d(y, N),$$

and so $\|y\| / d(y, N) \leq 1 + \epsilon$. But, since the space has finite dimension, and ϵ is arbitrary, we can obtain a convergent sequence $\{y_n\}$ such that $\|y_n\| = 1$, $y_n \rightarrow y$ and $\|y_n\| / d(y_n, N) \rightarrow 1$, whence $\|y\| = d(y, N)$.

Theorem 1.12

Let $T \in B(X, Y)$ have closed range and finite nullity. Then there exists $\epsilon > 0$ such that if $B \in B(X, Y)$ and $\|B\| < \epsilon$,

- (i) $T + B$ has closed range,
- (ii) $\text{nul}(T + B) \leq \text{nul } T$,
- (iii) $\text{def}(T + B) \leq \text{def } T$.

Proof:

We first prove (ii) making no use of the fact that $\text{Nul } T < \infty$.

Let $x \in (T + B)^{-1}(0)$, $x \neq 0$; then if $\|B\| < \gamma(T)$, we have

$$\gamma(T) d(x, T^{-1}(0)) \leq \|Tx\| = \|Bx\| \leq \|B\| \|x\| < \gamma(T) \|x\|.$$

Hence, for all non-zero $x \in (T + B)^{-1}(0)$,

$$d(x, T^{-1}(0)) < \|x\|,$$

so by Lemma 1.11, $\dim (T + B)^{-1}(0) \leq \dim T^{-1}(0)$ and (ii) is proved for $\epsilon \leq \gamma(T)$.

Now let $X = X_1 \oplus T^{-1}(0)$, where X_1 is closed, and let T_1, B_1 be the restrictions of T, B respectively to X_1 . Since $T_1 X_1 = TX$, T_1 has closed range, and so $\gamma(T_1) > 0$. Thus if $\|B\| < \gamma(T_1)$, so that $\|B_1\| < \gamma(T_1)$,

$$\|(T_1 + B_1)x\| \geq \|T_1 x\| - \|B_1 x\| \geq (\gamma(T_1) - \|B_1\|) \|x\|,$$

whence $T_1 + B_1$, and hence $T + B$, by Lemma 1.3, has closed range.

To prove (iii), we note that T^* has closed range, by Theorem 1.10, so $\gamma(T^*) > 0$ and if $\|B\| = \|B^*\| < \gamma(T^*)$, then $\text{nul}(T^* + B^*) \leq \text{nul } T^*$. The result follows from Theorem 1.10 and its Corollary on putting $\epsilon = \min(\gamma(T), \gamma(T_1), \gamma(T^*))$.

Lemma 1.13

Let $T \in \mathcal{B}(X, Y)$, with $\text{nul } T = 0$. Then there exists $\epsilon > 0$ such that if $B \in \mathcal{B}(X, Y)$ and $\|B\| < \epsilon$, $\text{def}(T + B) = \text{def } T$.

Proof:

T is one-to-one with closed range, so it has a bounded left inverse, S_0 . Let P be a bounded projection onto TX , and write $S = S_0 P$; S is bounded. Let $\epsilon = \|S\|^{-1}$. Then if $\|B\| < \epsilon$, $\|BS\| < 1$, so $U = I + BS$ is invertible in $\mathcal{B}(X)$, and

$$T + B = UT.$$

Since U is invertible and T has closed range, UT has closed range,

and, since U is one-to-one and onto, $\text{def } UT = \text{def } T$, so

$$\text{def } (T + B) = \text{def } T .$$

Theorem 1.14

Let T belong to $\Phi(X, Y)$; then there exists $\epsilon > 0$ such that if B belongs to $B(X, Y)$ and $\|B\| < \epsilon$, we have

- (i) $T + B$ is Fredholm,
- (ii) $\text{nul } (T + B) \leq \text{nul } T$, $\text{def } (T + B) \leq \text{def } T$,
- (iii) $\kappa(T + B) = \kappa(T)$.

Proof:

By Theorem 1.12, we have only to show (iii). We first establish a property of κ .

Let $S: X \rightarrow Y$ be a linear map, and let Z be a subspace of X , with $x \in X$ and Z_1 the linear span of x and Z ; we assume $x \notin Z$. There are two cases.

- (i) $Sx \in SZ \Rightarrow SZ_1 = SZ$ and $\text{nul } S|_{Z_1} = \text{nul } S|_Z + 1$,
 $\Rightarrow \kappa(S|_{Z_1}) = \kappa(S|_Z) + 1$ (if either side exists).
- (ii) $Sx \notin SZ \Rightarrow SZ_1$ spanned by SZ and Sx , but not by SZ ,
 $\Rightarrow \text{def } S|_{Z_1} = \text{def } S|_Z - 1$, $\text{nul } S|_{Z_1} = \text{nul } S|_Z$,
 $\Rightarrow \kappa(S|_{Z_1}) = \kappa(S|_Z) + 1$ (if either side exists).

Therefore, by induction, if Z_1 is an extension of Z such that Z is of codimension n in Z_1 ,

$$\kappa(S|_{Z_1}) = \kappa(S|_Z) + n .$$

Now let $X = X_1 \oplus T^{-1}(0)$, where X_1 is closed, with T_1, B_1 as before. Clearly $\text{def } T = \text{def } T_1$, and the above discussion shows that $\kappa(T + B) = \kappa(T_1 + B_1) + \text{nul } T$.

Moreover, by Theorem 1.12 and Lemma 1.13, there exists $\epsilon > 0$ such that if $\|B_1\| \leq \|B\| < \epsilon$, $T_1 + B_1$ is one-to-one with the same deficiency as T_1 , so

$$\begin{aligned}\kappa(T + B) &= \kappa(T_1 + B_1) + \text{nul } T \\ &= \text{nul } (T_1 + B_1) - \text{def } (T_1 + B_1) + \text{nul } T \\ &= \quad 0 \quad - \quad \text{def } T_1 \quad + \text{nul } T \\ &= \text{nul } T - \text{def } T \\ &= \kappa(T),\end{aligned}$$

and the Theorem is proved.

Theorem 1.15 (Gohberg & Krein)

Let $T, B \in B(X, Y)$, where T is Fredholm and X and Y are Banach spaces. Then there exists $\epsilon > 0$ such that $\text{nul } (T - \lambda B)$ is constant for $0 < |\lambda| < \epsilon$. Further, if $T \in B(X)$, and G is a component of $\ell\text{FSp } T$, then for every $\lambda \in G$, with the possible exception of some isolated points, $\text{nul } (T - \lambda I)$ has a constant value.

Proof:

Case (i): $\kappa(T) = 0$.

Let Z be a finite dimensional subspace of Y such that

$TX \oplus Z = Y$, which is possible since T is Fredholm. Since $\kappa(T) = 0$ $T^{-1}(0)$ and Z have the same finite dimension, so we can find a nonsingular linear map $U_0: T^{-1}(0) \rightarrow Z$. We extend U_0 to an element $U \in B(X, Y)$ by defining $U = U_0 P$, where P is a bounded projection of X onto $T^{-1}(0)$.

If we denote $T + U$ by T_1 we have $\text{nul } T_1 = \text{def } T_1 = 0$. By Theorem 1.12, $\exists \rho > 0$ such that $\text{nul } (T_1 - \lambda B) = \text{def } (T_1 - \lambda B) = 0$, that is $(T_1 - \lambda B)^{-1}$ exists for $|\lambda| < \rho$, and $R(\lambda) = (T_1 - \lambda B)^{-1}$ is a $B(X, Y)$ -valued analytic function of λ for $|\lambda| < \rho$. Also

$$\begin{aligned} (T - \lambda B)x = 0 &\iff (T_1 - \lambda B)x = Ux, \\ &\iff x = R(\lambda)Ux. \end{aligned} \quad (1)$$

We can write

$$Ux = \sum_{i=1}^n f_i(x) e_i,$$

where $n = \text{def } T$, $e_i \in Z$ and $f_i \in X^*$ ($i = 1, \dots, n$). Hence (1) holds if and only if

$$x = \sum_{i=1}^n R(\lambda) f_i(x) e_i.$$

Therefore, $\text{nul } (T - \lambda B)$, the number of linearly independent solutions of the equation

$$(T - \lambda B)x = 0,$$

is equal to the number of linearly independent solutions of the simultaneous equations

$$x = \sum_{i=1}^n \xi_i R(\lambda) e_i,$$

$$\xi_k = f_k(x) \quad (k = 1, \dots, n).$$

This is equal to the number of linearly independent solutions of the set of equations

$$\xi_k = \sum_{i=1}^n \xi_i f_k(R(\lambda) e_i) \quad (k = 1, \dots, n),$$

which may be written

$$\sum_{i=1}^n \{ \delta_{ik} - f_k(R(\lambda)e_i) \} \xi_i = 0 \quad (k = 1, \dots, n).$$

that is

$$A(\lambda)\xi = 0, \quad (2)$$

where $A(\lambda)$ is the $n \times n$ matrix whose (k, i) -th entry is $\delta_{ik} - f_k(R(\lambda)e_i)$, and ξ is the column vector (ξ_1, \dots, ξ_n) . (δ is here the Kronecker delta.)

The entries of $A(\lambda)$ are analytic functions of λ for $|\lambda| < \rho$, so $\det A(\lambda)$ is an analytic function of λ ($|\lambda| < \rho$), as are all its minors. If all the entries of $A(\lambda)$ are identically zero, the equation (2) has n linearly independent solutions for all values of λ with $|\lambda| < \rho$. If not all the entries of $A(\lambda)$ are identically zero, let p be the order of the largest minor $\Delta(\lambda)$ which is not identically zero (in the set $\{\lambda: |\lambda| < \rho\}$, of course). Since the zeros of analytic functions of one variable are isolated, $\exists \epsilon > 0$ such that $\Delta(\lambda)$ is non-zero in the annulus $\{\lambda: 0 < |\lambda| < \epsilon\}$. Thus for $0 < |\lambda| < \epsilon$ the largest minor of $A(\lambda)$ which is non-zero is of order p , and hence, for such values of λ , $A(\lambda)$ has rank p , and the equation (2) has $n - p$ linearly independent solutions.

We have just shown that, in either of the two cases considered, which exhaust the possibilities, there is a number $\epsilon > 0$ such that for $0 < |\lambda| < \epsilon$ the equation (2) has a constant number of linearly independent solutions. But we saw earlier that this equation has the same number of linearly independent solutions as the equation

$$(T - \lambda B)x = 0,$$

which therefore has a constant number of linearly independent solutions in the annulus, i.e., $\text{nul}(T - \lambda B)$ is constant in the annulus.

Case (ii): $\kappa(T) > 0$.

Let Z be any $\kappa(T)$ -dimensional subspace of Y and form the (exterior) direct sum $\tilde{Y} = Y \oplus Z$, and endow it with the norm

$$\| (y, z) \| = \| y \| + \| z \| \quad \forall y \in Y, z \in Z.$$

Define $\tilde{T}, \tilde{B} \in B(X, \tilde{Y})$ by $\tilde{T}x = (Tx, 0)$, $\tilde{B}x = (Bx, 0)$ so that $\kappa(\tilde{T}) = 0$, and we may apply case (i) to \tilde{T} . The result follows on observing that $\text{nul}(T - \lambda B) = \text{nul}(\tilde{T} - \lambda \tilde{B})$ for all values of λ .

Case (iii): $\kappa(T) < 0$.

This is proved analogously, by extending X to \tilde{X} by the addition of a subspace of dimension $|\kappa(T)|$; the nullities of the original and extended mappings now differ by $|\kappa(T)|$.

To prove the second part of the theorem, we first notice that if $\lambda_0 \in G$, a component of $\mathcal{EFS}p T$, then $T - \lambda I$ is Fredholm, and hence satisfies the postulates of the first part of the theorem. If we take I instead of B in this first part, we have shown the existence of $\epsilon > 0$ such that $\text{nul}(T - \lambda I)$ has a constant value in the annulus $\{\lambda: 0 < |\lambda - \lambda_0| < \epsilon\}$. Define $\psi(\lambda_0)$ to take this value; ψ is then a well-defined mapping from G to the set P of non-negative integers. If we endow P with the discrete topology, ψ is continuous; to see this, let $|\mu - \lambda_0| < \epsilon$, where λ_0 and ϵ are as above. Then $\{\lambda: 0 < |\lambda - \mu| < \min(|\lambda_0 - \mu|, \epsilon - |\lambda_0 - \mu|)\}$ is an annulus lying inside $\{\lambda: 0 < |\lambda - \lambda_0| < \epsilon\}$ on which $\text{nul}(T - \lambda I)$ has the constant value $\psi(\lambda_0)$, that is $\psi(\mu) = \psi(\lambda_0)$ in a neighbourhood of λ_0 and ψ is continuous. Therefore, since G is connected by hypothesis, $\psi(G)$ is connected, and so consists of a single point, say n .

Let λ_0 be a point of G for which $\text{nul}(T - \lambda I) \neq n$. Since $\psi(G) = \{n\}$, there exists $\epsilon > 0$ such that $\text{nul}(T - \lambda I) = n$ for λ in the range $0 < |\lambda - \lambda_0| < \epsilon$, that is, there exists a neighbourhood of λ_0 containing no other point of G for which $\text{nul}(T - \lambda I) \neq n$, and the theorem is proved.

Corollary:

Let G be a component of $\mathcal{E}\text{FSp } T$. If G contains a point λ_0 for which $T - \lambda_0 I$ is invertible, then, with the possible exception of some isolated points, $T - \lambda I$ is invertible for all λ in G . In particular, if G is the unbounded component of $\text{FSp } T$, then $G \cap \text{Sp } T$ contains only isolated points of $\text{Sp } T$, and may be void.

Proof:

If $T - \lambda_0 I$ is invertible, then $T - \lambda I$ is invertible for λ in some neighbourhood of λ_0 , so $\psi(\lambda_0) = 0$. Since ψ is constant on G , $\text{nul}(T - \lambda I) = 0$ for all λ in G with the possible exception of some isolated points.

Let $\phi: G \rightarrow P$ be defined as $\phi(\lambda) = \kappa(T - \lambda I)$. Theorem 1.14, part (iii) shows ϕ is continuous on $\mathcal{E}\text{FSp } T$, so $\phi(G)$ is connected and hence contains only one point. But, $T - \lambda_0 I$ is invertible, so $\text{nul}(T - \lambda_0 I) = \text{def}(T - \lambda_0 I) = 0$, and $\phi(\lambda_0) = 0$ so, for all λ belonging to G , $\text{nul}(T - \lambda I) = \text{def}(T - \lambda I)$. It follows that $T - \lambda I$ is invertible at all points of G for which $\text{nul}(T - \lambda I) = 0$ and the result follows by the previous paragraph.

The case where G is the unbounded component of $\mathcal{E}\text{FSp } T$ follows on observing that G must contain points which do not belong to the spectrum.

§ 4. Strictly singular operators

In this section X and Y will always denote infinite dimensional Banach spaces.

Definition

Let $T \in B(X, Y)$. T is said to be strictly singular if, for all infinite-dimensional subspaces M of X , $T|_M$ has no bounded left inverse.

Lemma 1.16

Let T belong to $B(X, Y)$ and suppose that T does not have closed range when restricted to any subspace of finite codimension in X . Then, $\forall \epsilon > 0$, there exists an infinite-dimensional subspace M_ϵ of X such that $T|_{M_\epsilon}$ is compact and has norm not exceeding ϵ .

Proof:

Since T does not have closed range when "restricted" to X itself, $\gamma(T) = 0$, so $\forall \epsilon > 0$, $\exists x_1$ with $\|x_1\| = 1$, and $\|Tx_1\| < \frac{1}{3}\epsilon$. Choose f_1 in X^* such that $\|f_1\| = f_1(x_1) = 1$. Suppose that we have found pairs $(x_1, f_1), \dots, (x_n, f_n)$ such that

$$\begin{aligned} \|f_k\| &= \|x_k\| = f_k(x_k) = 1 & (k = 1, \dots, n) \\ f_k(x_j) &= 0 & (k < j) \\ \|Tx_k\| &< \left(\frac{1}{3}\right)^k \epsilon & (k = 1, \dots, n) \end{aligned} \tag{1}$$

Now $N_n = \bigcap_{k=1}^n f_k^{-1}(0)$ is of finite codimension in X , so $T|_{N_n}$ does not have closed range, that is $\gamma(T|_{N_n}) = 0$ and $\exists x_{n+1}$ in N_n such that $\|x_{n+1}\| = 1$ and $\|Tx_{n+1}\| < \left(\frac{1}{3}\right)^{n+1} \epsilon$. Choose f_{n+1} such that $\|f_{n+1}\| = f_{n+1}(x_{n+1}) = 1$. The pairs $(x_1, f_1), \dots, (x_{n+1}, f_{n+1})$ now satisfy the conditions (1), so, by induction, we

can construct an infinite sequence of such pairs. Let M be the linear span of $\{x_n\}_{n=1}^{\infty}$; M has infinite dimension. Let x be an arbitrary member of M ; x is of the form

$$x = \sum_{i=1}^n \xi_i x_i, \quad (2)$$

for some integer n , and complex numbers ξ_1, \dots, ξ_n . Then

$$|\xi_1| = |f_1(x)| \leq \|x\|.$$

We make the induction hypothesis that

$$|\xi_k| \leq 2^{k-1} \|x\|, \quad (k = 1, \dots, r).$$

Then we have, in turn,

$$\begin{aligned} \xi_{r+1} &= f_{r+1}(x) - \sum_{i=1}^r \xi_i f_{r+1}(x_i), \\ |\xi_{r+1}| &\leq \|x\| + \sum_{i=1}^r |\xi_i| \\ &\leq \|x\| (1 + \sum_{i=1}^r 2^{i-1}) \\ &= 2^r \|x\|. \end{aligned}$$

Hence,

$$|\xi_k| \leq 2^{k-1} \|x\| \quad (k = 1, \dots, r+1),$$

so, by induction, this holds for all positive integers r .

Therefore, we have, for all x in M ,

$$\|Tx\| \leq \sum_{i=1}^n |\xi_i| \|Tx_i\| \leq \sum_{i=1}^n 2^{i-1} \left(\frac{1}{3}\right)^i \epsilon \|x\| < \epsilon \|x\|,$$

and $\|T|M\| \leq \epsilon$.

It only remains to show that $T|M$ is compact, which we shall do by showing that $T|M$ is the norm limit of finite rank operators.

Let $T_n: M \rightarrow Y$ be defined by

$$T_n x_k = \begin{cases} T x_k & \text{if } k = 1, \dots, n, \\ 0 & \text{if } k > n, \end{cases}$$

and extend by linearity to M . T_n is clearly of finite rank, and bounded. Let x belong to M and write x in the form (2), giving

$$\begin{aligned} \|Tx - T_m x\| &\leq \sum_{i=m+1}^{\infty} |\xi_i| \|Tx_i\| \\ &\leq \epsilon \|x\| \sum_{i=m+1}^{\infty} 2^{i-1} \left(\frac{1}{3}\right)^i \\ &< \left(\frac{2}{3}\right)^m \epsilon \|x\|, \end{aligned}$$

so $\|T_m - T\| \rightarrow 0$ as $m \rightarrow \infty$, and the theorem is proved.

Corollary:

If TX is not closed, then $\forall \epsilon > 0$, \exists an infinite-dimensional subspace M_ϵ such that $T|_{M_\epsilon}$ is compact with norm at most ϵ .

Proof:

Let N be a subspace of finite codimension in X ; write X as $X = N \oplus L$, where L is of finite dimension. Then TN is not closed, for if it were, by Lemma 1.3, TX would be closed, since TN is of finite codimension in TX . But N was any subspace of finite codimension in X , so the hypotheses of Lemma 1.16 hold, and the result follows.

Theorem 1.17

If T belongs to $B(X, Y)$, the following are equivalent:-

- (i) T is strictly singular,
- (ii) For every infinite-dimensional closed subspace M , there is an infinite-dimensional closed subspace N of M such that $T|_N$ is compact,
- (iii) For all $\epsilon > 0$ and for all infinite-dimensional closed subspaces $M \subseteq X$, there exists an infinite-dimensional ^{closed} subspace N of M such that $\|T|_N\| \leq \epsilon$.

Proof:

(i) \Rightarrow (ii): Since T is strictly singular, $T|_M$ has no bounded left inverse, where M is any closed infinite-dimensional subspace of X . If $M \cap T^{-1}(0)$ is of infinite dimension, put $N = M \cap T^{-1}(0)$; if not, let L be a closed complement in M of $M \cap T^{-1}(0)$. L has infinite dimension, so $T|_L$ has no bounded left inverse, whence TL is not closed, since $T|_L$ is one-to-one. Therefore TM is not closed. The Corollary to Lemma 1.16, applied to the map $T|_M$ now shows the existence of an infinite-dimensional subspace N_0 of M such that $T|_{N_0}$ is compact. Since the closure of N_0 is a subspace of M , and $T|_{N_0}$ is compact, the required subspace may be taken to be the closure of N_0 .

(ii) \Rightarrow (iii): Let M be any closed infinite-dimensional subspace of X . If $M \cap T^{-1}(0)$ has infinite dimension, we may put $N = M \cap T^{-1}(0)$ and then $\|T|_N\| = 0 < \epsilon$. If $M \cap T^{-1}(0)$ has finite dimension, let L be a closed complement of $M \cap T^{-1}(0)$ in M . L has infinite dimension, and $T|_L$ is one-to-one. By hypothesis, there exists an infinite-dimensional subspace K of L such that $T|_K$ is compact, so $T|_L$ cannot have a bounded left inverse since this would mean that $T|_K$ had a bounded left inverse, which is absurd. Therefore, $T|_L$ does not have closed range, so $T|_M$ does not have closed range, and we may apply the Corollary to Lemma 1.16 to $T|_M$ to obtain an infinite-dimensional subspace N_0 such that $\|T|_{N_0}\| \leq \epsilon$, and, on letting N be the closure of N_0 , we have the desired result.

(iii) \Rightarrow (i): It is clear that T cannot have a bounded left inverse on any closed infinite dimensional subspace if (iii) holds.

We show that (in our case where X is a Banach space) this implies that T is strictly singular. Let N be any infinite-dimensional subspace of X such that $T|N$ has a bounded left inverse, say $S: TN \rightarrow X$. Then $STx = x \quad \forall x \in N$. Extend S to \overline{TN} , the closure of TN as follows. If $Tx_n \rightarrow y$, then $\{STx_n\}$ is a Cauchy sequence in N and so contains a subsequence which converges to an element x of \overline{N} ; define $Sy = x$. S is well defined, for let $x_n \rightarrow x$, $z_n \rightarrow z$ as $n \rightarrow \infty$, where $x_n, z_n \in N$ ($n = 1, 2, \dots$), be such that $Tx_n \rightarrow y$ and $Tz_n \rightarrow y$. Then

$$\begin{aligned} x - z &= \lim_{n \rightarrow \infty} (x_n - z_n) \\ &= \lim_{n \rightarrow \infty} S(Tx_n - Tz_n) \\ &= 0, \end{aligned}$$

since S is bounded. It follows that S extends to \overline{TN} and that T has a bounded left inverse when restricted to \overline{N} , a closed infinite-dimensional subspace, which we deduced above was impossible. It follows that T cannot have a bounded inverse on any infinite-dimensional subspace, that is, that T is strictly singular, and the Theorem is proved.

Corollary:

The set of strictly singular operators in $B(X)$ forms a closed two-sided ideal.

Proof:

Let T, U be strictly singular, and let M be any closed infinite-dimensional subspace of X . By Theorem 1.17, there exists a closed infinite-dimensional subspace N_1 of M such that $T|N_1$ is

compact; by the same result, there is an infinite-dimensional closed subspace N_2 of N_1 such that $U|_{N_2}$ is compact. Therefore there exists a subspace N_2 of M such that $(T + U)|_{N_2}$ is compact, and statement (ii) of Theorem 1.17 holds, so $T + U$ is strictly singular.

We show that TS is strictly singular, where T is strictly singular and $S \in B(X)$. Assume that TS has a bounded inverse when restricted to some subspace M . Then $\exists \kappa > 0$ such that

$$\|TSx\| \geq \kappa \|x\| \geq \kappa \frac{\|Sx\|}{\|S\|} \quad \forall x \in M,$$

hence T has a bounded left inverse on SM , which must therefore have finite dimension. But $S|_M$ is one-to-one by the first inequality so M has finite dimension if SM has. Hence TS is strictly singular. A similar argument shows ST is strictly singular.

Clearly, if T is strictly singular and $\lambda \in \mathbb{C}$, λT is strictly singular.

Now let $T_n \rightarrow T$, where T_n is strictly singular ($n = 1, 2, \dots$). Assume that T has a bounded left inverse on some subspace M ; then $\exists \kappa > 0$ such that

$$\|Tx\| \geq \kappa \|x\| \quad \forall x \in M.$$

Let n_0 be sufficiently large that

$$\|T_n - T\| < \frac{1}{2}\kappa \quad \forall n > n_0.$$

Then, for $n > n_0$,

$$\begin{aligned} \|T_n x\| &\geq \left| \|Tx\| - \|(T_n - T)x\| \right| \\ &\geq \kappa \|x\| - \frac{1}{2}\kappa \|x\| \\ &= \frac{1}{2}\kappa \|x\| \quad \forall x \in M. \end{aligned}$$

Hence, since T_n is strictly singular, M has finite dimension and T is strictly singular.

Theorem 1.18 (Kato)

If H is a Hilbert space and $S \in B(H)$, then S is strictly singular if and only if S is compact.

Proof:

We need only show that a strictly singular operator S is compact. We first show that if $\lambda \neq 0$, $S - \lambda I$ has closed range and finite nullity. Let $0 < \epsilon < |\lambda|$, and let M be a subspace such that $\|(S - \lambda I)|_M\| \leq \epsilon$. Then, for x in M ,

$$\|Sx\| \geq \|\lambda x\| - \|(S - \lambda I)x\| \geq (|\lambda| - \epsilon)\|x\|,$$

so M has finite dimension; in particular, $\text{nul}(S - \lambda I) < \infty$. If $S - \lambda I$ has non-closed range, the Corollary to Theorem 1.16 ensures an infinite-dimensional subspace M such that $\|(S - \lambda I)|_M\| \leq \frac{1}{2}|\lambda|$, which contradicts the conclusion above; hence $(S - \lambda I)X$ is closed.

By the polar decomposition theorem, there is a partial isometry U such that $S = U|S|$, $|S| = U^*S$ where $|S| = (S^*S)^{\frac{1}{2}}$, so we need only show that a self-adjoint strictly singular operator is compact; we shall assume that $S = S^*$. Since $\text{nul}(S - \lambda I) = \text{def}(S - \lambda I)$ for $\lambda \in \mathbb{R} \setminus \{0\}$ (for $S = S^*$), and $\text{Sp } S \subseteq \mathbb{R}$, we see that, for all $\lambda \neq 0$, $\text{nul}(S - \lambda I) = \text{def}(S - \lambda I) < \infty$. Hence $S - \lambda I \in \Phi(H)$ for all $\lambda \neq 0$ and so $\text{FSp } S = \{0\}$. Let $\tau: B(H) \rightarrow B(H)/K(H)$ be the canonical map. Then since $\tau(S)$ is self-adjoint in $B(H)/K(H)$

$$\|\tau(S)\| = \rho(\tau(S)) = 0,$$

and so S is compact.

CHAPTER 2

THE CAPACITY OF AN ELEMENT OF A BANACH ALGEBRA

§ 1. Definitions and fundamentals.

In his paper "Capacity in Banach Algebras", [6], Halmos introduces the idea of capacity in the following way: let a be an element of a normed algebra A over the field F (where F is either the real or complex numbers), with identity element 1 .

Define

$$\text{cap}_n a = \inf \{ \|p(a)\| : p \text{ a monic polynomial of degree } n \text{ over } F \},$$

and

$$\text{cap } a = \inf_n (\text{cap}_n a)^{\frac{1}{n}}.$$

The quantity $\text{cap } a$ is called the capacity of a .

Since a product of monic polynomials is monic, we have

$$\text{cap}_{m+n} a \leq \text{cap}_m a \cdot \text{cap}_n a,$$

whence it follows that

$$\text{cap } a = \inf_n (\text{cap}_n a)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} (\text{cap}_n a)^{\frac{1}{n}}.$$

Clearly this definition of capacity is independent of the norm up to equivalence. In the case of a Banach algebra, the notion of capacity is connected in a simple way with the capacity of plane sets; this will be shown in §2.

Let P_n denote the set of monic polynomials (that is, those whose leading coefficient is 1) of degree n over F , where F is the field of the algebra with which we are concerned.

Lemma 2.1

There exists $p \in P_n$ for which $\|p(a)\| = \text{cap}_n a$.

Proof:

If $p \in P_n$, then $\exists \underline{\alpha} = (\alpha_0, \dots, \alpha_{n-1}) \in \mathbb{C}^n$ such that

$$p(a) = \alpha_0 1 + \dots + \alpha_{n-1} a^{n-1} + a^n.$$

We introduce the auxiliary functions $\theta, \phi: \mathbb{C}^n \rightarrow \mathbb{R}$, where

$$\theta(\underline{\alpha}) = \| \alpha_0 1 + \dots + \alpha_{n-1} a^{n-1} + a^n \|,$$

$$\phi(\underline{\alpha}) = \| \alpha_0 1 + \dots + \alpha_{n-1} a^{n-1} \|.$$

Clearly θ and ϕ are continuous, and

$$\text{cap}_n a = \inf \{ \theta(\underline{\alpha}) : \underline{\alpha} \in \mathbb{C}^n \}.$$

Let $|\cdot|$ be a norm on \mathbb{C}^n , and write $S_r = \{ \underline{\alpha} : |\underline{\alpha}| = r \}$, and

$$\gamma = \inf \{ \phi(\underline{\alpha}) : \underline{\alpha} \in S_1 \}.$$

If $\gamma = 0$, then since S_1 is compact and ϕ continuous, ϕ attains its infimum, i.e. $\exists \alpha' \in S_1$ with

$$\alpha'_0 1 + \dots + \alpha'_{n-1} a^{n-1} = 0.$$

Since $(0, \dots, 0) \notin S_1$, we can, by multiplication by a scalar and a suitable power of a , obtain from this a polynomial $p \in P_n$ with $p(a) = 0$, and the infimum used in defining $\text{cap}_n a$ is attained.

If, on the other hand, $\gamma \neq 0$, then $\phi(\underline{\alpha}) \geq r\gamma \quad \forall \underline{\alpha} \in S_r$.

Choosing r so large that $r\gamma > \text{cap}_n a + 1 + \|a^n\|$, we have

$$\begin{aligned} \theta(\underline{\alpha}) &\geq \phi(\underline{\alpha}) - \|a^n\| \\ &\geq r\gamma - \|a^n\| \quad \forall \underline{\alpha} \in S_r \\ &> \text{cap}_n a + 1 \quad \forall \underline{\alpha} \in S_r. \end{aligned}$$

Hence the closed set E given by

$$E = \{ \underline{\alpha} : \theta(\underline{\alpha}) \leq \text{cap}_n a + 1 \}$$

satisfies the equation $S_r \cap E = \emptyset$. As this holds for all r sufficiently large, E is bounded and hence compact. But

$$\text{cap}_n a = \inf \{ \theta(\underline{\alpha}) : \underline{\alpha} \in E \},$$

so the infimum is attained, and there exists α for which

$$\text{cap}_n a = \| \alpha_0 1 + \dots + \alpha_{n-1} a^{n-1} + a^n \| ,$$

and the polynomial p where $p(z) = \alpha_0 + \dots + \alpha_{n-1} z^{n-1} + z^n$ has the required properties.

The definition of the capacity of an element of a normed algebra may be extended to algebras without identity in two natural ways, that is, by adjoining an identity and using the existing definition or by making a new definition taking infima over monic polynomials without constant term. We show that these give rise to the same quantity.

Let P'_n be the set of monic polynomials of degree n over the appropriate field without constant term, and write

$$\text{cap}'_n a = \inf \{ \| p(a) \| : p \in P'_n \} ,$$

and

$$\text{cap}' a = \lim_{n \rightarrow \infty} (\text{cap}_n a)^{\frac{1}{n}} = \inf_n (\text{cap}_n a)^{\frac{1}{n}} .$$

Proposition 2.2

If A is a normed algebra with an identity 1 , then

$$\text{cap } a = \text{cap}' a \quad \forall a \in A .$$

Proof:

$$\begin{aligned} \text{cap}'_n a &= \inf \{ \| p(a) \| : p \in P'_n \} \\ &\leq \inf \{ \| p(a) \| : p \in P_n \} \\ &= \text{cap}_n a . \end{aligned}$$

Let $\text{cap}_n a = \| p(a) \|$, where $p \in P_n$; this is possible by Lemma 2.1. Define $q \in P'_{n+1}$ by $q(z) = zp(z) \quad \forall z \in \mathbb{C}$. We then have in turn,

$$\text{cap}'_{n+1} a \leq \| q(a) \| \leq \| a \| \cdot \| p(a) \| ,$$

$$(\text{cap}'_{n+1} a)^{\frac{1}{n+1}} \leq \| a \|_{n+1}^{\frac{1}{n+1}} (\text{cap}_n a)^{\frac{1}{n+1}},$$

$$\text{cap}' a \leq \text{cap} a.$$

From this result it follows that if the algebra Λ does not possess an identity, we may adjoin one and the value of the capacity obtained will be independent of the way in which this is done. Alternatively, we may consider $\text{cap}' a$, to arrive at the same quantity. Moreover, since any algebra with a 1 may be equivalently renormed so that $\| 1 \| = 1$ and capacity is independent of norm up to equivalence, we may assume, for the purposes of evaluating capacity, that the algebra concerned is unital.

§ 2. Spectral capacity

There is a well-developed and venerable theory of the capacity of plane sets: for an exhaustive treatment see Tsuji [16], while sufficient for the purposes in hand may be found in Hille [7, Chapter 16].

For a compact subset E of \mathbb{C} , we make the definitions

$$\text{Cap}_n E = \inf \{ \| p \|_E : p \in P_n \},$$

$$\text{Cap} E = \lim_{n \rightarrow \infty} (\text{Cap}_n E)^{\frac{1}{n}} = \inf_n (\text{Cap}_n E)^{\frac{1}{n}},$$

where $\| p \|_E = \sup \{ |p(z)| : z \in E \}$.

Notice that if we consider the algebra $C(E)$, then the function a , where $a(z) = z \ \forall z \in E$, lies in $C(E)$, and we may apply our previous definition of capacity to it. Clearly $\text{cap}_n a = \text{Cap}_n E$, for all n , whence Lemma 2.1 shows the existence of $p \in P_n$ for which $\| p \|_E = \text{Cap}_n E$.

It is obvious from the definitions above that $E \subseteq F$ implies that

$\text{Cap } E \leq \text{Cap } F$, if E, F are both compact subsets of \mathbb{C} .

Lemma 2.3

If A and B are compact subsets of \mathbb{C} whose symmetric difference (i.e. the set $(A \setminus B) \cup (B \setminus A)$) is a finite set, then A and B have the same capacity.

Proof:

It is enough to show that if $B = A \cup \{\lambda\}$, then A and B have the same capacity.

Clearly, $\text{Cap } A \leq \text{Cap } B$. Let $p_n \in P_n$ ($n = 1, 2, \dots$) be such that $\text{Cap}_n A = \| p_n \|_A$. Now define $q_n \in P_{n+1}$ as

$$q_n(z) = (z - \lambda)p_n(z) ,$$

and let $d = \sup\{|z - \lambda| : z \in A\}$. If $d = 0$ the result is trivial, so assume $d > 0$. Then

$$\text{Cap}_{n+1} B \leq \| q_n \|_B = \| q_n \|_A \leq d \cdot \text{Cap}_n A ,$$

which gives

$$\begin{aligned} \text{Cap } B &= \lim_{n \rightarrow \infty} (\text{Cap}_{n+1} B)^{\frac{1}{n+1}} \\ &\leq \lim_{n \rightarrow \infty} \left(d^{\frac{1}{n+1}} (\text{Cap}_n A)^{\frac{1}{n+1}} \right) \\ &= \text{Cap } A . \end{aligned}$$

At this stage we pause to make a number of observations. Firstly, since

$$\text{cap}_n a = \inf \{ \| p(a) \| : p \in P_n \} \leq \| a^n \| \quad \forall a \in A ,$$

we have

$$\text{cap } a \leq \rho(a) \quad \forall a \in A .$$

Let $\lambda \neq 0$; then if

$$p(z) = z^n + \alpha_{n-1}z^{n-1} + \dots + \alpha_0 ,$$

write

$$p(\lambda; z) = z^n + \lambda^{-1} \alpha_{n-1} z^{n-1} + \dots + \alpha_0 \lambda^{-n}.$$

Clearly p , $p(\lambda; \cdot)$ both lie in P_n . We have

$$\begin{aligned} \text{cap}_n \lambda a &= \inf \{ \|p(\lambda a)\| : p \in P_n \} \\ &= \inf \{ \|\lambda^n p(\lambda; a)\| : p \in P_n \} \\ &\geq |\lambda|^n \text{cap}_n a \quad \forall a \in A, \lambda \in \mathbb{C} \setminus \{0\}. \end{aligned}$$

Hence, writing $a = \lambda^{-1} \lambda a$ and using the same inequality,

$$\text{cap}_n \lambda a = |\lambda|^n \text{cap}_n a,$$

and, since the case for $\lambda = 0$ is trivial, we have, for all $\lambda \in \mathbb{C}$,

$$\text{cap} \lambda a = |\lambda| \text{cap} a.$$

From the definition of the capacity of a plane set, a finite set has capacity zero, since we can find a monic polynomial of arbitrarily large degree which is identically zero on the set. An analogue for elements of a normed algebra is as follows:

Lemma 2.4(Halmos)

If p is a non-zero polynomial, and $p(a)$ has zero capacity, then $\text{cap} a = 0$.

Proof:

Since $\text{cap} \lambda p(a) = |\lambda| \text{cap} p(a)$ for all $\lambda \in \mathbb{C}$, we may assume that p is monic. Let p have degree n . Since

$$\text{cap}_k p(a) = \inf \{ \|q(p(a))\| : q \in P_k \},$$

and $q \circ p \in P_{nk}$, we have

$$\text{cap}_{nk} a = \inf \{ \|r(a)\| : r \in P_{nk} \} \leq \text{cap}_k p(a).$$

Hence

$$\text{cap} a = \inf_m (\text{cap}_m a)^{\frac{1}{m}} \leq \inf_k \{ (\text{cap}_k p(a))^{\frac{1}{k}} \}^{\frac{1}{n}} = 0,$$

since n is fixed and $\text{cap} p(a) = 0$.

Theorem 2.5 (Halmos)

If A is a unital Banach algebra, then

$$\text{cap } a = \text{Cap Sp } a \quad \forall a \in A.$$

Proof:

First we consider the complex case.

Let $p \in P_n$; then

$$\begin{aligned} \|p\|_{\text{Sp } a} &= \sup \{ |p(\lambda)| : \lambda \in \text{Sp } a \} \\ &= \sup \{ |\lambda| : \lambda \in p(\text{Sp } a) \} \\ &= \rho(p(a)) \\ &\leq \|p(a)\|. \end{aligned}$$

Therefore

$$\begin{aligned} \text{Cap}_n \text{ Sp } a &= \inf \{ \|p\|_{\text{Sp } a} : p \in P_n \} \\ &\leq \inf \{ \|p(a)\| : p \in P_n \} \\ &= \text{cap}_n a, \end{aligned}$$

and

$$\text{Cap Sp } a \leq \text{cap } a. \quad (*)$$

By Lemma 2.1, $\exists s_k \in P_k$ ($k = 1, 2, \dots$) for which we have

$$\begin{aligned} \|s_k\|_{\text{Sp } a} &= \text{Cap}_k \text{ Sp } a, \text{ so } \lim_k \{ \|s_k\|_{\text{Sp } a} \}^{\frac{1}{k}} = \text{Cap Sp } a. \text{ But} \\ \|s_k\|_{\text{Sp } a} &= \rho(s_k(a)) = \lim_k \| \{s_k(a)\}^n \|^{\frac{1}{n}}. \end{aligned}$$

If $\|s_k\|_{\text{Sp } a} = 0$ for some k , then $\rho(s_k(a)) = 0$ and $\text{cap } s_k(a) = 0$. Lemma 2.4 now shows that $\text{cap } a = 0$, whence, by (*), we have

$$\text{Cap Sp } a = \text{cap } a = 0.$$

If $\|s_k\|_{\text{Sp } a} \neq 0$ ($k = 1, 2, \dots$), there exists a positive integer $n = n(k)$ such that, for $n \geq n(k)$,

$$\| \{s_k(a)\}^n \|^{\frac{1}{n}} \leq 2 \|s_k\|_{\text{Sp } a},$$

Then $q \in P_n$ and $\|q\|_{Sp\ a} \leq \|p_0\|_{Sp\ a}$, so

$$\|q\|_{Sp\ a} = \inf \{ \|p\|_{Sp\ a} : p \in P_n \}.$$

But q is a real polynomial, so we may assume, if required, that

a polynomial $s_k \in P_k$ for which

$$\|s_k\|_{Sp\ a} = \text{Cap}_k\ Sp\ a$$

is real.

This will be used also in Theorem 2.6.

Corollary:

If A is any Banach algebra,

$$\text{cap } a = \text{Cap } Sp\ a \quad \forall a \in A.$$

Proof:

We adjoin an identity to A to produce A_1 ; let \underline{a} be the canonical image of a in A_1 , where a is any element of A . By Proposition 2.2, $\text{cap } a = \text{cap } \underline{a}$, while $Sp\ a \setminus Sp\ \underline{a} \subseteq \{0\}$ and $Sp\ \underline{a} \subseteq Sp\ a$ ([11, p28]), so Lemma 2.3 shows that $\text{Cap } Sp\ a = \text{Cap } Sp\ \underline{a}$, and Theorem 2.5 completes the proof.

At this point, a brief summary of some of the properties of the capacity of plane sets is desirable; for proofs see [7] and [16].

The capacity of plane sets already defined in terms of the minimum of norms of polynomials is known also as the Tchebycheff constant of the set. It is equal to the transfinite diameter, which is formulated as follows: let

$$\{\delta_n(E)\}^{\frac{1}{2n(n-1)}} = \sup \left\{ \prod_{1 \leq i < j \leq n} |z_i - z_j| : z_1, \dots, z_n \in E \right\},$$

then $\delta_n(E)$ decreases to a limit $\delta(E)$, the transfinite diameter, as $n \rightarrow \infty$. Yet another way of obtaining the same quantity associated

with a set is the logarithmic capacity. This we shall not use explicitly, so we do not give its definition, but we notice that the logarithmic capacity is only defined for compact sets (though it can be extended to Borel sets), whereas the other two are well-defined for any bounded subset of \mathbb{C} . Use of the logarithmic capacity allows the conclusion that if E_n is a compact set in the plane ($n = 1, 2, \dots$) such that $\text{Cap } E_n = 0$ ($n = 1, 2, \dots$) and $E = \bigcup_{n=1}^{\infty} E_n$ is compact, then $\text{Cap } E = 0$.

A moment's thought now shows that any countable compact set has zero capacity, as it is the union of countably many sets consisting of a single point.

From these somewhat assorted remarks, we observe that the set of elements of a Banach algebra which have capacity zero includes those with finite and countable spectra, e.g. idempotents, quasinilpotents, and in the case of the algebra $B(X)$, for a Banach space X , compact and Riesz operators.

§ 3. Quasialgebraic elements and operators.

We have seen that any element of a Banach algebra which has a finite spectrum has zero capacity. Since this includes those elements which satisfy a polynomial identity, in other words, the algebraic elements, we can regard capacity as a measure of how "far" an element is from being algebraic. For this reason we follow Halmos and call elements which have zero capacity quasialgebraic.

It was not difficult to show (Lemma 2.4) that if $p(a)$ is quasialgebraic for some non-zero polynomial p , then a is also quasialgebraic. What is not so obvious is the following theorem of

Halmos. We assume that A is a Banach algebra over F and that a is an element of A .

Theorem 2.6(Halmos)

If a is quasia algebraic, and p is any polynomial over F , then $p(a)$ is quasia algebraic.

Proof:

First assume A is a complex algebra, and assume without loss that p is monic. We will show that $\text{Cap } p(\text{Sp } a) = 0$, which is enough since $p(\text{Sp } a) = \text{Sp } p(a)$.

Let $s_n \in P_n$ be such that $\|s_n\|_{\text{Sp } a} = \text{Cap}_n \text{Sp } a$, ($n = 1, 2, \dots$) Since $\text{cap } a = 0$, $\|s_n\|_{\text{Sp } a}^{\frac{1}{n}} \rightarrow 0$ as $n \rightarrow \infty$. Factorise s_n , to obtain

$$s_n(z) = \{z - \lambda_1^{(n)}\} \dots \{z - \lambda_n^{(n)}\} \quad \forall z \in \mathbb{C},$$

and put

$$q_n(z) = \{z - p(\lambda_1^{(n)})\} \dots \{z - p(\lambda_n^{(n)})\}.$$

Since $z - \lambda_j^{(n)}$ divides $p(z) - p(\lambda_j^{(n)})$ ($j = 1, 2, \dots, n$), s_n divides $q_n \circ p$, so write

$$q_n(p(z)) = s_n(z) \cdot r_n(z) \quad \forall z \in \mathbb{C}, n = 1, 2, \dots$$

If $r_n(\lambda) = 0$, then $p(\lambda) = p(\lambda_j^{(n)})$ for some $j \in \{1, 2, \dots, n\}$, so $\lambda \in p^{-1}\{p(\lambda_j^{(n)})\}$. Now the zeros of the Tchebycheff polynomials of a set E (that is the polynomials which attain the value $\text{Cap}_n E$ and are monic of degree n) lie in $\overline{\text{co}} E$, the closed convex hull of E , hence $\lambda_j^{(n)} \in \overline{\text{co}} \text{Sp } a$ and the zeros of r_n lie in $p^{-1}\{p(\overline{\text{co}} \text{Sp } a)\}$. This is a compact set since $\overline{\text{co}} \text{Sp } a$ is compact and p is a polynomial.

Write $d = \sup \{|\mu| : \mu \in p^{-1}[p(\overline{\text{co}} \text{Sp } a)]\}$, and let p have degree k , so that r_n is of degree $(k-1)n$ and its zeros lie in

the disc of radius d centred on 0 , therefore

$$\| r_n \|_{Sp\ a} \leq (2d)^{(k-1)n}$$

since $Sp\ a$ also lies in this disc. Hence

$$\| q_n \circ p \|_{Sp\ a} = \| r_n s_n \|_{Sp\ a} \leq \| s_n \|_{Sp\ a} (2d)^{(k-1)n}.$$

But,

$$\| q_n \circ p \|_{Sp\ a} = \| q_n \|_{p(Sp\ a)},$$

so we have

$$\text{Cap}_n\ p(Sp\ a) \leq \| q_n \|_{p(Sp\ a)} \leq \| s_n \|_{Sp\ a} (2d)^{(k-1)n},$$

and

$$\text{Cap}\ p(Sp\ a) = 0,$$

since $\| s_n \|_{Sp\ a}^{\frac{1}{n}} \rightarrow 0$ as $n \rightarrow \infty$, and the theorem is proved in the complex case.

If, on the other hand, A is a real algebra, and p consequently a real polynomial, we must be more circumspect; the above reasoning will remain true provided we can show that all the polynomials constructed are real, since we consider only real polynomials acting on real algebras. Since s_n ^{can be taken} ~~is~~ real, the roots $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ are either real or occur in complex conjugate pairs, hence the roots $p(\lambda_1^{(n)}), \dots, p(\lambda_n^{(n)})$ of q_n are either real or occur in conjugate pairs, so q_n is a real polynomial. It follows that $q_n \circ p$ is a real polynomial, and since $q_n \circ p = r_n s_n$, r_n must be a real polynomial. The remainder of the proof proceeds as before.

Remark: The above theorem was proved in the context of a Banach algebra but it is true in the case of a normed algebra. Let A be a normed algebra and let \hat{A} be its completion, with $a \rightarrow \hat{a}$ the canonical homomorphism of A into \hat{A} ; this is an isometry, so for all

polynomials p and all $a \in A$, $\|p(a)\| = \|p(\tilde{a})\| = \|(p(\tilde{a}))\|$. Therefore $\text{cap } a = \text{cap } \tilde{a}$ and $\text{cap } p(a) = \text{cap } p(\tilde{a})$. If a is quasia-
algebraic, so is \tilde{a} and Theorem 2.6 shows $p(\tilde{a})$ is quasia-
algebraic, hence $p(a)$ is also.

It is elementary that if a and b both satisfy polynomial identities and $ab = ba$, then $a + b$ and ab are also algebraic, though the result need not be true if a and b do not commute. To take this one stage further, consider the algebra $B(l^1)$ and let A, B be operators in $B(l^1)$ defined as follows: we denote by e_k the element of l^1 whose k -th entry is 1 and whose other entries are all 0. Let

$$Ae_k = \begin{cases} e_{k+1} & (k \text{ even}) \\ 0 & (k \text{ odd}), \end{cases}$$

$$Be_k = \begin{cases} 0 & (k \text{ even}) \\ e_{k+1} & (k \text{ odd}). \end{cases}$$

Then $A^2 = B^2 = 0$ and $AB \neq BA$. Therefore A and B are nilpotent and so algebraic, but $A + B$ is the unilateral shift, and it is easily seen that

$$\text{cap}_n (A + B) = \inf \{ \|p(A + B)\| : p \in P_n \} = 1,$$

so $\text{cap} (A + B) = 1$, and $A + B$ is not even quasia-
algebraic.

The case where a and b are two commuting quasia-
algebraic elements of a Banach algebra is tantalising; it is natural to ask whether the sum of two such elements is quasia-
algebraic, but the answer is not known. The best that can be done at the moment is:

Theorem 2.7

Let a, b be two commuting quasia-
algebraic elements of a Banach

algebra A , and let $\text{Sp } a$ be countable. Then $a + b$ and ab are quasialgebraic.

Proof:

Let B be a maximal closed commutative subalgebra of A containing a and b ; then ([11], Theorem 1.6.14) $\text{Sp}_A x = \text{Sp}_B x$ for all $x \in B$, where $\text{Sp}_B x$ denotes the spectrum of x considered as an element of B . By considering the expression of the spectrum in terms of the carrier space of B , we have

$$\text{Sp } (a + b) \subseteq \text{Sp } a + \text{Sp } b = \{ \lambda + \mu : \lambda \in \text{Sp } a, \mu \in \text{Sp } b \}.$$

Further, $\text{Sp } a + \text{Sp } b$ is a compact subset of \mathbb{C} , since if $\{ \lambda_n + \mu_n \}$ is any sequence of points in $\text{Sp } a + \text{Sp } b$, with $\lambda_n \in \text{Sp } a$, $\mu_n \in \text{Sp } b$ ($n = 1, 2, \dots$), then the (sequential) compactness of $\text{Sp } a$ and $\text{Sp } b$ shows the existence of a subsequence $n_1(k)$ such that $\lambda_{n_1(k)}$ converges to an element of $\text{Sp } a$, and a subsequence $n_2(k)$ of $n_1(k)$ such that $\mu_{n_2(k)}$ converges to an element of $\text{Sp } b$, so that $\lambda_{n_2(k)} + \mu_{n_2(k)}$ converges to an element of $\text{Sp } a + \text{Sp } b$.

Now

$$\text{Sp } a + \text{Sp } b = \bigcup_{\lambda \in \text{Sp } a} (\lambda + \text{Sp } b),$$

and the set $\lambda + \text{Sp } b$, being a translate of $\text{Sp } b$, has zero capacity, so the set $\text{Sp } a + \text{Sp } b$ is the union of a countable number of sets of zero capacity, and so itself has capacity zero, whence $\text{cap } (a + b) = 0$.

The result for ab follows similarly from the inclusion

$$\text{Sp } ab \subseteq (\text{Sp } a)(\text{Sp } b) = \{ \lambda \mu : \lambda \in \text{Sp } a, \mu \in \text{Sp } b \}.$$

Remark: This result can also be extended to normed algebras. Let A be a normed algebra, \hat{A} its completion. We shall regard \hat{A} as the set of equivalence classes of Cauchy sequences of A , where the equivalence relation \sim is defined as: $\{a_n\} \sim \{b_n\}$ iff $a_n - b_n \rightarrow 0$ as $n \rightarrow \infty$. Denote equivalence classes by $[\cdot]$ and define $\|[\{a_n\}]\| = \lim_n \|a_n\|$. In all this $a_n, b_n \in A$ ($n = 1, 2, \dots$). For $a \in A$, let $\hat{a} = [\{a_n\}]$ where $a_n = a$ ($n = 1, 2, \dots$). Then the map $a \rightarrow \hat{a}$ is an isometric homomorphism, and therefore preserves capacity, so the result will be proved if we show that $\text{Sp } \hat{a}$ is countable if $\text{Sp } a$ is. But $a \rightarrow \hat{a}$ is an identity-preserving homomorphism (if an identity exists), and so if a is invertible (quasiregular) so is \hat{a} hence $\text{Sp } \hat{a} \subseteq \text{Sp } a$, for all $a \in A$.

Notice in this treatment that no attempt is made to evaluate the capacity of the spectrum of an element of a normed algebra; these spectra display pathologies (such as unboundedness) which do not lend themselves to such treatment.

The following counterexample shows that, while the problem of whether or not the sum of two quasialgebraic elements is itself quasialgebraic remains unanswered, the stronger conjecture that capacity is subadditive or submultiplicative for commuting elements is false. Let

$$A = \{z \in \mathbb{C} : \min(|z - 10|, |z + 10|) \leq 1\},$$

$$X = \{(\xi_\lambda)_{\lambda \in \Lambda} : \xi_\lambda \in \mathbb{C}, \sum_{\lambda \in \Lambda} |\xi_\lambda| < \infty\}.$$

X is a Banach space. Define A and B in $B(X)$ as follows: let e_μ be the element of X given by (ξ_λ) where $\xi_\mu = 1$ and $\xi_\lambda = 0$ for $\lambda \in \Lambda \setminus \{\mu\}$. Then write

$$Ae_{\lambda} = \begin{cases} (\lambda - 10)e_{\lambda} & \text{if } \operatorname{Re} \lambda > 0, \lambda \in \Lambda \\ (\lambda + 10)e_{\lambda} & \text{if } \operatorname{Re} \lambda < 0, \lambda \in \Lambda, \end{cases}$$

$$Be_{\lambda} = \begin{cases} 10e_{\lambda} & \text{if } \operatorname{Re} \lambda > 0, \lambda \in \Lambda \\ -10e_{\lambda} & \text{if } \operatorname{Re} \lambda < 0, \lambda \in \Lambda, \end{cases}$$

and extend by linearity to X . Then $\operatorname{Sp} A = \{z: |z| \leq 1\}$ and $\operatorname{Sp} B = \{-10, 10\}$ but $\operatorname{Sp} (A + B) = \Lambda$. We must estimate $\operatorname{Cap} \Lambda$; using the transfinite diameter we have

$$\begin{aligned} \{\delta_n(\Lambda)\}^{\frac{1}{2}n(n-1)} &= \sup \left\{ \prod_{1 \leq i < j \leq n} |z_i - z_j| : z_1, \dots, z_n \in \Lambda \right\} \\ &= \max_{0 \leq k \leq n} \sup \left\{ \prod_{1 \leq i < j \leq n} |z_i - z_j| : z_1, \dots, z_k \in \Lambda_+, z_{k+1}, \dots, z_n \in \Lambda_- \right\}, \end{aligned}$$

where $\Lambda_+ = \{\lambda \in \Lambda: \operatorname{Re} \lambda > 0\}$, $\Lambda_- = \{\lambda \in \Lambda: \operatorname{Re} \lambda < 0\}$.

Since $|\lambda_1 - \lambda_2| \geq 18$ if $\lambda_1 \in \Lambda_+$, $\lambda_2 \in \Lambda_-$, and $|\mu_1 - \mu_2| \leq 1$ if μ_1 and μ_2 are both in Λ_+ or both in Λ_- , we take $n = 2m$, and consider m points in each of Λ_+ and Λ_- , to obtain

$$\{\delta_{2m}(\Lambda)\}^{m(2m-1)} \geq \{\delta_m(\Lambda_+)\}^{\frac{1}{2}m(m-1)} \{\delta_m(\Lambda_-)\}^{\frac{1}{2}m(m-1)} (18)^{m^2},$$

where the points in Λ_+ , Λ_- have been chosen so that the appropriate product of terms containing only points of one of them attains its maximum value of $\{\delta_m(\Lambda)\}^{\frac{1}{2}m(m-1)}$. Since Λ_+ , Λ_- are discs of radius 1, they have capacity 1, so that $\delta_m(\Lambda_+) \rightarrow 1$ as $m \rightarrow \infty$, and

$$\lim \delta_{2m}(\Lambda) \geq \lim 1 \cdot 1 \cdot (18)^{\frac{m^2}{2m-1}} = 3\sqrt{2},$$

that is $\operatorname{Cap} \Lambda \geq 3\sqrt{2}$. Hence

$$\operatorname{cap} (A + B) \geq 3\sqrt{2} > 1 = \operatorname{cap} A + \operatorname{cap} B,$$

so capacity is not subadditive for commuting elements. A similar construction shows the corresponding result for products.

The set of quasialgebraic elements of a Banach algebra need not be closed, for let $\Lambda = L_{\infty}([0, 1])$ with pointwise multiplication,

and let $f_n, f \in A$ be given by

$$f_n(t) = \frac{[nt]}{n} \quad \text{a.e. in } [0, 1],$$

$$f(t) = t \quad \text{a.e. in } [0, 1],$$

for $n = 1, 2, \dots$, where $[x]$ denotes the integral part of x , that is the integer k for which $k \leq x < k + 1$. Now it is immediate that $\text{Sp } f_n = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$ and $\text{Sp } f = [0, 1]$, also that $f_n \rightarrow f$ as $n \rightarrow \infty$ since

$$\|f_n - f\| = \sup |f_n(t) - f(t)| \leq \frac{1}{n}.$$

But $\text{cap } f_n = 0$ ($n = 1, 2, \dots$) and $\text{cap } f = \frac{1}{4}$, and the result is proved.

The powers of an element behave somewhat better, however, and we have the inequality

$$\text{cap } a^n \geq (\text{cap } a)^n \quad (n = 1, 2, \dots).$$

To see this we notice that if $p \in P_k$, then for some $\alpha_0, \dots, \alpha_{k-1}$

$$\begin{aligned} p(a^n) &= (a^n)^k + \alpha_{k-1}(a^n)^{k-1} + \dots + \alpha_0 \\ &= a^{nk} + \alpha_{k-1}a^{n(k-1)} + \dots + \alpha_0 \\ &= q(a), \end{aligned}$$

where $q \in P_{nk}$. From this we have $\text{cap}_k a^n \geq \text{cap}_{nk} a$, and

$$\begin{aligned} \text{cap } a^n &\geq \lim_k \{(\text{cap}_{nk} a)^{\frac{1}{n}}\}^n \\ &= (\text{cap } a)^n. \end{aligned}$$

This inequality can be strict; let a be an element of a Banach algebra with $\text{Sp } a = [0, 1]$. Then $\text{Sp } a^n = [0, 1]$ and so $\text{cap } a^n = \frac{1}{4}$ but $(\text{cap } a)^n = (\frac{1}{4})^n < \frac{1}{4}$ if $n > 1$.

Quasialgebraic operators are related to compressions and dilations in a surprising way. In the terminology of Sz.Nagy and Foias [17], if H is a subspace of a Hilbert space K , we write

$S = \text{pr } T$ if $S = P_H T|_H$ where P_H is the projection onto H . S is then said to be the compression of T to H . If $S \in B(H)$ and $T \in B(K)$ are such that $S^n = \text{pr } T^n$ ($n = 1, 2, \dots$) we say that T is a dilation of S .

If T is a dilation of S , then $\text{cap } S \leq \text{cap } T$, for if p is a polynomial and $S^n = \text{pr } T^n$ ($n = 1, 2, \dots$)

$$\begin{aligned} p(S) &= \alpha_n S^n + \dots + \alpha_0 I \\ &= \{\alpha_n P_H T^n + \dots + \alpha_0 P_H\}|_H \\ &= P_H p(T)|_H, \end{aligned}$$

so $\|p(S)\| \leq \|p(T)\|$ and $\text{cap } S \leq \text{cap } T$.

On the other hand, if S is a compression of T , no such relation need hold. For example let K be the Hilbert space spanned by two orthonormal sequences $\{e_k\}$ and $\{f_k\}$ where each member of one sequence is orthogonal to all the members of the other, say

$$K = \left\{ \sum_{k=1}^{\infty} \xi_k e_k + \eta_k f_k : \sum_{k=1}^{\infty} |\xi_k|^2 + |\eta_k|^2 < \infty \right\}$$

with $\left\| \sum_{k=1}^{\infty} \xi_k e_k + \eta_k f_k \right\|^2 = \sum_{k=1}^{\infty} |\xi_k|^2 + |\eta_k|^2$. Let H be the subspace $H = \{x \in K : \eta_k = 0, (k = 1, 2, \dots)\}$ where x denotes the typical member of K . Define $T \in B(K)$ by

$$T e_k = e_{k+1} + f_{k+1} \quad (k = 1, 2, \dots)$$

$$T f_k = -e_{k+1} - f_{k+1} \quad (k = 1, 2, \dots).$$

Clearly $T^2 = 0$, so T is nilpotent, and hence quasia algebraic, but the compression of T to H is the unilateral shift, which is certainly not quasia algebraic.

Before leaving this section we digress to state some results of Halmos, which, while undoubtedly part of the theory of capacity, are not relevant to what follows. Proofs will be found in Halmos [5,6] and Douglas and Pearcy [1].

An operator A on a Hilbert space H is said to be quasitriangular if there is a net $\{P_\alpha: \alpha \in \Lambda\}$ of finite rank projections converging strongly to I , for which $\|P_\alpha A P_\alpha - A P_\alpha\| \rightarrow 0$. In the case of a separable space, this reduces to the following: A is quasitriangular if there is an orthonormal basis with respect to which A has the matrix $[a_{ij}]$ where (i) $a_{ij} = 0$ if $i > j + 1$, and (ii) $\{a_{i+1,i}\}$ has a subsequence converging to zero.

By reducing the situation to the case where A has a cyclic vector e , by a theorem of Douglas and Pearcy, so that we can orthonormalise the set $\{e, Ae, A^2e, \dots\}$ to obtain a basis where A has matrix $[a_{ij}]$ with $a_{ij} = 0$ if $i > j + 1$, and then consideration of the $(j + n, j)$ th entry of $p(A)$ for a polynomial p , Halmos shows that if there exists a sequence $\{p_n\}$ of monic polynomials with p_n of degree n , such that $\|p_n(A)\|^{1/n} \rightarrow 0$, then

$$\left\{ \prod_{1 \leq k \leq n} a_{j+k, j+k-1} \right\}^{1/n} \rightarrow 0 \text{ as } n \rightarrow \infty \quad (j = 1, 2, \dots),$$

whence $\{a_{i+1,i}\}$ contains a subsequence converging to zero. Hence we have

Theorem 2.8(Halmos)

A quasia algebraic operator on a separable Hilbert space is quasitriangular.

Corollary(Halmos)

Every operator on a separable Hilbert space with countable spectrum is quasitriangular.

This Corollary solves a problem raised by Douglas and Pearcy. Quasitriangular operators are of interest in the invariant subspace

problem, but we shall not deal further with them here. The interested reader is referred to the papers of Halmos [5] and Douglas and Percy [1].

§ 4. Perturbations.

As a preliminary to this section, we observe the fact that the capacity of any compact subset of the complex plane equals that of its outer boundary, i.e. the intersection of the set with the closure of the unbounded component of its complement. This is proved in the sources quoted for capacity; as we shall make some use of the fact, we introduce some notation.

If A is a compact subset of \mathbb{C} , then \mathcal{A} has at most countably many components, exactly one of which is unbounded. Denote the unbounded one by $A^{(0)}$ and the others by $A^{(1)}, A^{(2)}, \dots$, adopting the convention that $A^{(n)} = A^{(n+1)} = \dots = \emptyset$ if there are only n components. The set $\mathcal{A}^{(0)}$ is compact and has the same outer boundary as A and hence the same capacity, but is simply connected.

Definition

Let X be a Banach space, $T \in B(X)$. We define the essential spectrum, $\text{ess } T$, of T to be the set

$$\text{ess } T = \text{FSp } T \cup (\text{Sp } T)'$$

where $'$ denotes the derived set, i.e. the set of accumulation points.

In this entire section, we shall be dealing only with Banach spaces X , unless otherwise mentioned; as the results are trivial otherwise we shall assume throughout that X is an infinite-dimensional Banach space.



Lemma 2.9

Let $T \in B(X)$. Then $\text{ess } T$ and $\text{FSp } T$ have the same outer boundary and hence the same capacity.

Proof:

$(\text{FSp } T)^{(0)}$ is the unbounded component of $\text{FSp } T$, so it certainly contains a point λ_0 for which $T - \lambda_0 I$ is invertible. By the Corollary to Theorem 1.15, it follows that $T - \lambda I$ is invertible for all $\lambda \in (\text{FSp } T)^{(0)}$ with the possible exception of some isolated points. That is, if $\lambda \in (\text{FSp } T)^{(0)} \cap \text{Sp } T$, then there exists $\delta > 0$ for which

$$0 < |\lambda - \lambda'| < \delta \Rightarrow T - \lambda' I \text{ is invertible,}$$

$$\Rightarrow \lambda \notin \text{Sp } T,$$

$$\text{whence } \lambda \notin (\text{Sp } T)',$$

so $(\text{FSp } T)^{(0)} \cap (\text{Sp } T)' = \emptyset$, and since

$$\text{FSp } T \cap (\text{FSp } T)^{(0)} = \emptyset,$$

we have

$$(\text{FSp } T)^{(0)} \cap \text{ess } T = \emptyset,$$

or

$$(\text{FSp } T)^{(0)} \subseteq \text{ess } T.$$

But $(\text{FSp } T)^{(0)}$ is an unbounded connected set, hence

$$(\text{FSp } T)^{(0)} \subseteq (\text{ess } T)^{(0)}.$$

On the other hand, we have in turn,

$$\text{FSp } T \subseteq \text{ess } T,$$

$$\text{ess } T \subseteq \text{FSp } T,$$

$$(\text{ess } T)^{(0)} \subseteq (\text{FSp } T)^{(0)},$$

and the result is proved.

Lemma 2.10

If A and B are compact subsets of \mathbb{C} with $B' \subseteq A \subseteq B$, then A and B have the same capacity.

Proof:

Let $\epsilon > 0$, so there exists n and p , where p is a monic polynomial of degree n , such that

$$\|p\|_A^{\frac{1}{n}} = (\text{Cap}_n A)^{\frac{1}{n}} < \text{Cap } A + \epsilon.$$

If we put

$$G_n = \{z \in \mathbb{C} : |p(z)| < (\text{Cap } A + \epsilon)^n\},$$

we obtain an open neighbourhood G_n of A , so there is a compact set K_n for which $A \subseteq \text{int } K_n \subseteq K_n \subseteq G_n$. Hence,

$$\text{Cap}_n K_n \leq \|p\|_{K_n}^{\frac{1}{n}} < (\text{Cap } A + \epsilon)^n,$$

and

$$\text{Cap } K_n = \inf_m (\text{Cap}_m K_n)^{\frac{1}{m}} \leq \text{Cap } A + \epsilon.$$

Now $B \cap \text{int } K_n$ is a compact set whose accumulation points lie in A , i.e. outside $B \cap \text{int } K_n$, hence it has no accumulation points and is therefore a finite set. It follows that $B \cap \text{int } K_n$ is also finite, so that by Lemma 2.3,

$$\text{Cap } B = \text{Cap } B \cap \text{int } K_n \leq \text{Cap } K_n \leq \text{Cap } A + \epsilon.$$

But ϵ was arbitrary, and $A \subseteq B$ so we have

$$\text{Cap } A = \text{Cap } B.$$

Theorem 2.11

Let X be a Banach space and J any closed two-sided ideal of $B(X)$, such that J contains only Riesz operators. Then if

$\pi: B(X) \rightarrow B(X)/J$ is the canonical epimorphism,

$$\text{cap } T = \text{cap } \pi(T) \quad \forall T \in B(X).$$

Proof:

By Theorem 1.7, we know that $\text{Sp } \pi(T) = \text{FSp } T$, while, since $B(X)/J$ is a unital Banach algebra, Theorem 2.5 shows that it is sufficient to show that $\text{Sp } T$ and $\text{FSp } T$ have the same capacity. But $(\text{Sp } T)' \subseteq \text{ess } T \subseteq \text{Sp } T$, so Lemma 2.10 shows that $\text{ess } T$ and $\text{Sp } T$ have the same capacity, and an application of Lemma 2.9 (equality of $\text{Cap FSp } T$ and $\text{Cap ess } T$) completes the proof.

Corollary:

If $T \in B(X)$ and $K \in J$, where J is any closed two-sided ideal of $B(X)$ contained in the Riesz operators, then

$$\text{cap } (T + K) = \text{cap } T .$$

The most important closed two-sided ideal of $B(X)$ which contains only Riesz operators is the ideal $K(X)$ of compact operators; in some spaces this is the only such ideal, but in general this is not so — the strictly singular operators form another such ideal, for example.

The Corollary to Theorem 2.11 leads us to ask whether the equation $\text{cap } (T + K) = \text{cap } T \quad \forall T \in B(X)$ holds for a wider class of operators K . In particular, is it true for all Riesz operators K ? The answer is negative, and the counterexample on p44 gives two nilpotent operators whose sum has positive capacity. Since nilpotent operators are Riesz, this establishes that perturbation by an arbitrary Riesz operator may alter capacity. If, however, the Riesz operator and the original operator commute, Theorem 2.7 shows that perturbation by such a Riesz operator leaves the capacity invariant.

The question remains, therefore, whether perturbation by some

larger class of operators than those already obtained leaves the capacity invariant; in particular, are there any closed two-sided ideals not contained in the set of Riesz operators for which the Corollary still holds? In the case of Hilbert space, the answer is negative, and we shall in fact prove rather more. We first need a lemma.

Lemma 2.12

Let H be a Hilbert space of infinite dimension. There exists $T \in B(H)$ for which $\text{cap } T > 0$.

Proof:

Let $\{e_\lambda: \lambda \in \Lambda\}$ be an orthonormal basis for H , and select from this set a countable subset $\{e_k: k = 1, 2, \dots\}$, which will itself form an orthonormal sequence. Define T by

$$T\left(\sum_{\lambda \in \Lambda} \xi_\lambda e_\lambda\right) = \sum_{k=1}^{\infty} \xi_k e_{k+1},$$

where we have identified the integer k with $\lambda \in \Lambda$ iff $e_k = e_\lambda$.

T is well-defined, linear and maps into H . Further

$$\left\| T\left(\sum_{\lambda \in \Lambda} \xi_\lambda e_\lambda\right) \right\|^2 = \sum_{k=1}^{\infty} |\xi_k|^2 \leq \sum_{\lambda \in \Lambda} |\xi_\lambda|^2,$$

so $T \in B(H)$. Moreover, if p is a polynomial, say

$$p(z) = z^n + \alpha_{n-1}z^{n-1} + \dots + \alpha_0 \quad \forall z \in \mathbb{C},$$

then

$$\begin{aligned} \|p(T)e_k\|^2 &= \|e_{k+n} + \alpha_{n-1}e_{k+n-1} + \dots + \alpha_0 e_k\|^2 \\ &\geq 1 + |\alpha_{n-1}| + \dots + |\alpha_0| \\ &\geq 1, \end{aligned}$$

so that $\text{cap}_n T = 1$, ($n = 1, 2, \dots$) and hence $\text{cap } T = 1$.

It is worth noticing at this point that it is an open question whether or not $B(X)$ contains an operator of positive capacity, where

X is an arbitrary Banach space. For most of the classical spaces the answer is clear - where it has meaning, a unilateral shift, or some analogy to one provides an example - but the general case is considerably more elusive, because of the possible scarcity of bounded operators on X , other than finite-rank and compact operators, which, of course, all have zero capacity.

Theorem 2.13

Let H be a Hilbert space. Any (left, right, two-sided) ideal of $B(H)$ which contains only quasia algebraic operators is contained in $K(H)$.

Proof:

Let J be a right ideal of $B(H)$ such that J contains a non-compact operator, T . By a result of Kato (Theorem 1.18), the set of strictly singular operators in $B(H)$ coincides with the set of compact operators, so T is not strictly singular. Therefore there exists a subspace M of H , of infinite dimension, on which T has a bounded left inverse. Let $TM = L$, and denote the left inverse of T by S , so that $S \in B(L, M)$ and

$$STx = x \quad \forall x \in M, \quad TSy = y \quad \forall y \in L.$$

By the continuity of S and T and the completeness of H , we may extend S to the closures of L ; we may therefore assume, without loss, that M and L are closed. Let P be the orthogonal projection onto L .

Let $R = SP$, so $R \in B(H)$ and $P = TR \in J$, since J is a right ideal, and $T \in J$. But if $U \in B(L)$, $PUP \in J$ since $P \in J$ and we have $Sp_{B(H)} PUP \supseteq Sp_{B(L)} U$, for

$$\begin{aligned}
 \lambda \notin \text{Sp}_{B(H)} PUP &\implies \lambda \notin \text{Sp}_{B(H)} UP \\
 &\implies \exists V \in B(H) \text{ such that } VUP - \lambda V = UPV - \lambda V = I, \\
 &\implies VUx - \lambda Vx = x \quad \forall x \in L, \text{ and } UPV - \lambda V = I, \\
 &\implies VUx - \lambda Vx = x \quad \forall x \in L, \text{ and} \\
 &\qquad \qquad \qquad \lambda Vx = UPVx - x \in L \quad \forall x \in L \\
 &\qquad \qquad \qquad \text{so } V(L) \subseteq L, \\
 &\implies VUx - \lambda Vx = x \quad \forall x \in L, \\
 &\qquad \qquad \qquad UVx - \lambda Vx = x \quad \forall x \in L \text{ since } V(L) \subseteq L, \\
 &\implies \lambda \notin \text{Sp}_{B(L)} U.
 \end{aligned}$$

Now an infinite-dimensional subspace of a Hilbert space is itself a Hilbert space if it is closed, so L is a Hilbert space, and Lemma 2.12 shows the existence of $U \in B(L)$ such that $\text{cap } U > 0$, hence there is a $U \in B(L)$ for which $\text{cap } PUP > 0$, which contradicts the fact that $PUP \in J$, since J contains only quasialebraic operators; hence J cannot contain any non-compact operators.

Now let J be a left ideal of $B(H)$ containing only quasialebraic operators. Then $J^* = \{T^* : T \in J\}$ is a right ideal, where $*$ denotes the usual Hilbert space adjoint. Since $\text{cap } T = \text{cap } T^* \quad \forall T \in B(H)$, because $T \rightarrow T^*$ is an isometric anti-automorphism of $B(H)$, J^* contains only quasialebraic operators, and the first part of the proof shows that $J^* \subseteq K(H)$, and hence $J \subseteq K(H)$, and the theorem is proved.

CHAPTER 3

JOINT CAPACITY

§ 1. Introduction.

It is natural to ask whether the concept of capacity may be extended to r -tuples of elements of a Banach algebra in a similar way to that of spectrum and numerical range. Since the capacity is, in a sense, a measure of the amount by which an element fails to be algebraic, we shall want to deal with polynomials in r variables. The theory of polynomials in several variables differs radically between the cases where the variables are assumed to commute and where they do not, so we shall restrict ourselves to the case where the elements of the r -tuple commute with each other; for this we may, without loss, assume that the r -tuple is composed of elements of a commutative Banach algebra.

We shall require a quantity akin to $\text{cap}_n a$, where a is a single element of a Banach algebra, and to do this we must find a replacement for the monic polynomials in one variable. Several contenders for the title spring to mind, of which we shall have to choose the most suitable, but first we require some notation.

Denote by \underline{k} an r -tuple of non-negative integers (k_1, \dots, k_r) and write $|\underline{k}| = k_1 + \dots + k_r$. Let $\underline{z} = (z_1, \dots, z_r) \in \mathbb{C}^r$, and

$$\underline{z}^{\underline{k}} = z_1^{k_1} z_2^{k_2} \dots z_r^{k_r}.$$

Any polynomial in r variables over \mathbb{C} of degree n is an expression of the form

$$p(z_1, \dots, z_r) = \sum_{|\underline{k}| \leq n} \alpha_{\underline{k}} \underline{z}^{\underline{k}},$$

where $\alpha_{\underline{k}} \in \mathbb{C}$ for all \underline{k} .

What then is to be "monic"? Since the definition is to concern the terms of highest degree, write

$$\mu(p) = \sum_{|\underline{k}|=n} |\alpha_{\underline{k}}|,$$

where p is defined as above. We shall define a polynomial p in r variables to be monic if $\mu(p) = 1$, and denote the set of such polynomials of degree n by $P_n(r)$.

This definition is, clearly, a normalisation of the polynomials in r variables, and any non-zero polynomial is a constant multiple of an element of $P_n(r)$, for appropriate n and r . Moreover, $P_n(1)$ consists of all polynomials in one variable whose leading coefficient is of modulus 1, which differs from the usual definition only by the possible multiplication by a constant of modulus 1.

It is elementary that if p and q are polynomials in r variables, $\mu(pq) \leq \mu(p)\mu(q)$. Since we are going to consider, as in the case of one variable, $\inf\{\|p(a_1, \dots, a_r)\| : p \in P_n(r)\}$, it is of fundamental interest to know by how much $\mu(pq)$ can be less than $\mu(p)\mu(q)$. The evaluation of this is also of interest in the theory of diophantine equations, and some estimates for μ have been obtained by workers in this field, and we shall start with these in the next section.

Before proceeding, however, we observe and reject another plausible definition of "monic". Let p be as before, and write

$$\nu(p) = \sum_{|k|=n} a_k.$$

Clearly $\nu(pq) = \nu(p)\nu(q)$, so a product of polynomials with $\nu(p) = 1$, for each polynomial p would also have $\nu(\text{product}) = 1$, which would eliminate the numerical factors which will arise with the other definition. We do not make use of this quantity ν in defining "monic", since it would not give rise to a normalisation of the polynomials, and non-zero polynomials exist which would not be constant multiples of "monic" ones (i.e. the polynomials p with

$\nu(p) = 0$), so any joint capacity concept involving only "monic" polynomials would bear no relation to such polynomials.

§ 2. Basic definitions and properties.

Let p be a polynomial in one variable, and denote by $M(p)$ the quantity

$$M(p) = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |p(e^{ix})| dx \right\}.$$

Lemma 3.1 (Mahler [9])

If p is a polynomial in one variable, say,

$$p(z) = \sum_{k=0}^n \alpha_k z^k,$$

then

$$|\alpha_k| \leq \binom{n}{k} M(p) \quad (k = 0, 1, \dots, n).$$

Proof:

Jensen's formula ([13]) states that if F is analytic in the interior and continuous on the boundary of the disc $\{z \in \mathbb{C}: |z| \leq \rho\}$ and $F(0) \neq 0$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |F(\rho e^{ix})| dx = \log |F(0)| + \sum_{\nu=1}^n \log \frac{\rho}{|\xi_\nu|},$$

where ξ_1, \dots, ξ_n are the zeros of F in the disc $\{z \in \mathbb{C}: |z| \leq \rho\}$ counted as many times as their multiplicities.

Now let p be the polynomial given, and assume for the present that $\alpha_0 \neq 0$; we may clearly assume throughout that $\alpha_n \neq 0$. Let the roots of p be ξ_1, \dots, ξ_n where

$$|\xi_1| \leq |\xi_2| \leq \dots \leq |\xi_n| \leq 1 < |\xi_{n+1}| \leq \dots \leq |\xi_n|.$$

Putting $\rho = 1$ in Jensen's formula gives

$$\log M(p) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{ix})| dx = \log |\alpha_0| + \sum_{\nu=1}^n \log \frac{1}{|\xi_\nu|}$$

$$\begin{aligned}
 &= \log \left| \frac{\alpha_0}{\xi_1 \xi_2 \dots \xi_N} \right| \\
 &= \log | \alpha_n \xi_{N+1} \dots \xi_n | , \tag{1}
 \end{aligned}$$

since $(-1)^n \alpha_0 = \alpha_n \xi_1 \dots \xi_n$. From the numbering of the zeros, it is clear that if $\{i_1, \dots, i_m\}$ is a set of m distinct integers from the set $\{1, 2, \dots, n\}$, we have

$$| \xi_{i_1} \dots \xi_{i_m} | \leq | \xi_{N+1} \dots \xi_n | ,$$

whence equation (1) shows that $| \alpha_n \xi_{i_1} \dots \xi_{i_m} | \leq M(p)$.

Now for $m = 0, 1, \dots, n-1$,

$$\begin{aligned}
 (-1)^m \alpha_{n-m} &= \sum_{\alpha_n} (\text{product of roots } m \text{ at a time}) \\
 &= \sum \xi_{i_1} \dots \xi_{i_m} ,
 \end{aligned}$$

where this last sum is taken over all possible sets $\{i_1, \dots, i_m\}$ of distinct integers in the set $\{1, 2, \dots, n\}$; there are $\binom{n}{m}$ such sets.

Hence

$$\begin{aligned}
 | \alpha_{n-m} | &= \sum | \alpha_n \xi_{i_1} \dots \xi_{i_m} | \leq \max | \alpha_n \xi_{i_1} \dots \xi_{i_m} | \binom{n}{m} \\
 &\leq \binom{n}{m} M(p) .
 \end{aligned}$$

The case for $m = 0$ is just Jensen's formula applied to p , so we have proved the result when $\alpha_0 \neq 0$.

Now let $\alpha_0 = 0$, and assume that $\alpha_0 = \alpha_1 = \dots = \alpha_{r-1} = 0 \neq \alpha_r$, and write $p(z) = z^r q(z)$, where

$$q(z) = \sum_{k=0}^{n-r} \alpha_{r+k} z^k .$$

Now

$$\begin{aligned}
 \log M(p) &= \frac{1}{2\pi} \int_0^{2\pi} \log |p(e^{ix})| dx = \frac{1}{2\pi} \int_0^{2\pi} \log |e^{irx} q(e^{ix})| dx \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \log |q(e^{ix})| dx \\
 &= \log M(q) ,
 \end{aligned}$$

and by the preceding results applied to q , we obtain

$$|\alpha_{r+k}| \leq \binom{n-r}{k} M(q) \quad (k = 0, 1, \dots, n-r).$$

Further, on noting that, for $k + r \leq n$, $\binom{n-r}{k} \leq \binom{n}{k+r}$, we have

$$|\alpha_{r+k}| \leq \binom{n}{k+r} M(p) \quad (k = 0, 1, \dots, n-r),$$

so that

$$|\alpha_m| \leq \binom{n}{m} M(p) \quad (k = 0, 1, \dots, n),$$

since $\alpha_0 = \alpha_1 = \dots = \alpha_{r-1} = 0$, and the lemma is proved.

Now let $p(z_1, \dots, z_r)$ be a polynomial in r variables of total degree n . Define

$$M_r(p) = \exp \left\{ \frac{1}{(2\pi)^r} \int_0^{2\pi} dx_1 \dots \int_0^{2\pi} \log |p(e^{ix_1}, \dots, e^{ix_r})| dx_r \right\}.$$

Now write

$$p(z_1, \dots, z_r) = \sum_{k_1=1}^n p_{k_1}(z_2, \dots, z_r) z_1^{k_1},$$

where p_{k_1} is a polynomial in $r-1$ variables of degree at most $n - k_1$. By Lemma 3.1, if $z_2, \dots, z_r \in \mathbb{C}$ are fixed, so that p is considered as a polynomial in the single variable z_1 , we have

$$|p_{k_1}(z_2, \dots, z_r)| \leq \binom{n}{k_1} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |p(e^{ix_1}, z_2, \dots, z_r)| dx_1 \right\}.$$

Hence

$$\log |p_{k_1}(e^{ix_2}, \dots, e^{ix_r})| \leq \log \binom{n}{k_1} + \frac{1}{2\pi} \int_0^{2\pi} \log |p(e^{ix_1}, \dots, e^{ix_r})| dx_1,$$

and, integrating $r-1$ times over the interval $[0, 2\pi]$ and dividing by $(2\pi)^{r-1}$ we obtain

$$M_{r-1}(p_{k_1}) \leq \binom{n}{k_1} M_r(p). \quad (1)$$

We now continue the process, and express $p_{k_1}, \dots, p_{k_{m-1}}(z_m, \dots, z_r)$ in terms of powers of z_m and define

$$p_{k_1, \dots, k_{m-1}}(z_m, \dots, z_r) = \sum_{k_m=0}^n p_{k_1, \dots, k_m}(z_{m+1}, \dots, z_r) z_m^{k_m},$$

where pk_1, \dots, k_m is a polynomial, possibly zero, and we have

$$pk_1, \dots, k_m = \alpha_{k_1, \dots, k_m},$$

since pk_1, \dots, k_m is a constant. At this stage we notice that

we may apply Lemma 3.1 to a polynomial of degree not exceeding n

and obtain exactly the same result, for let the actual degree of the

polynomial be $m \leq n$, then we have $|\alpha_k| \leq \binom{n}{k} M(p)$, and so, since

$\binom{n}{k} \leq \binom{n}{k}$ we have $|\alpha_k| \leq \binom{n}{k} M(p)$. We shall make use of this in

what follows. We have

$$\begin{aligned} |pk_1, \dots, k_r| &\leq \binom{n}{k_r} M_1(pk_1, \dots, k_{r-1}) && (\text{Lemma 3.1}) \\ &\leq \binom{n}{k_r} \binom{n}{k_{r-1}} M_2(pk_1, \dots, k_{r-2}) && \text{by (1)} \\ &\vdots \\ &\leq \binom{n}{k_r} \dots \binom{n}{k_1} M_r(p). \end{aligned}$$

Hence

$$|\alpha_{k_1, \dots, k_r}| \leq \binom{n}{k_1} \dots \binom{n}{k_r} M_r(p). \quad (2)$$

If we now denote by $L(p)$ the sum of the moduli of the coefficients of p , that is

$$L(p) = \sum_{|k| \leq n} |\alpha_k|,$$

then (2) shows that

$$\begin{aligned} L(p) &= \sum_{k_1 + \dots + k_r \leq n} |\alpha_{k_1, \dots, k_r}| \\ &< \sum_{k_1 + \dots + k_r = 1}^n |\alpha_{k_1, \dots, k_r}| \\ &\leq 2^{nr} M_r(p). \end{aligned} \quad (3)$$

Moreover, $|p(e^{ix_1}, \dots, e^{ix_r})| \leq L(p)$, for all real values of x_1, \dots, x_r , so we have, by the definition of $M_r(p)$,

$$M_r(p) \leq L(p). \quad (4)$$

Theorem 3.2 (Mahler)

If p_1, \dots, p_k are polynomials in r variables of degrees n_1, \dots, n_k respectively, then

$$2^{-(n_1 + \dots + n_k)r} L(p_1) \dots L(p_k) \leq L(p_1 \dots p_k) \leq L(p_1) \dots L(p_k) .$$

Proof:

$$\begin{aligned} \log M_r(p_1 \dots p_k) &= \frac{1}{(2\pi)^r} \int_0^{2\pi} dx_1 \dots \int_0^{2\pi} \log |(p_1 \dots p_k)(e^{ix_1}, \dots, e^{ix_r})| dx_r \\ &= \sum_{j=1}^k \frac{1}{(2\pi)^r} \int_0^{2\pi} dx_1 \dots \int_0^{2\pi} \log |p_j(e^{ix_1}, \dots, e^{ix_r})| dx_r \\ &= \sum_{j=1}^k \log M_r(p_j), \end{aligned}$$

whence we have

$$M_r(p_1 \dots p_k) = M_r(p_1) \dots M_r(p_k) .$$

Therefore we have, by (3) and (4)

$$\begin{aligned} L(p_1) \dots L(p_k) &\leq 2^{n_1 r + \dots + n_k r} M_r(p_1) \dots M_r(p_k) \\ &= 2^{(n_1 + \dots + n_k)r} M_r(p_1 \dots p_k) \\ &\leq 2^{(n_1 + \dots + n_k)r} L(p_1 \dots p_k) . \end{aligned}$$

Since $L(pq)$ denotes the sum of the moduli of the coefficients of pq , it is clear that for polynomials p and q , $L(pq) \leq L(p)L(q)$, whence

$$L(p_1 \dots p_k) \leq L(p_1) \dots L(p_k) ,$$

so that

$$2^{-(n_1 + \dots + n_k)r} L(p_1) \dots L(p_k) \leq L(p_1 \dots p_k) \leq L(p_1) \dots L(p_k) .$$

Corollary 1:

Let $p \in P_n(r)$, $q \in P_m(r)$. Then there exists a complex number λ for which $\lambda pq \in P_{m+n}(r)$; for such λ , we have $1 \leq |\lambda| \leq 2^{(m+n)r}$.

Proof:

Let p be the polynomial

$$p(z_1, \dots, z_r) = \sum_{|k| \leq n} a_k z^k,$$

and denote by p' the polynomial consisting only of the terms of highest degree, that is

$$p'(z_1, \dots, z_r) = \sum_{|k| = n} a_k z^k.$$

Define q' similarly; since p and q are monic of degrees n and m respectively, we know that p' and q' are not zero.

The polynomial pq is of degree at most $m+n$, and any terms of degree $m+n$ will be formed as sums of products of terms of degree n from p , and of degree m from q , that is, all terms of degree $m+n$ occur in the product $p'q'$. Since $p'q'$ contains only terms of degree $m+n$, it follows that $p'q'$ is the polynomial consisting of all the terms of pq of degree $m+n$, and so we have $(pq)' = p'q'$, provided pq contains terms of degree $m+n$, that is, provided $p'q'$ is not the zero polynomial. But p and q are monic, so, by definition, $L(p') = L(q') = 1$, and Theorem 3.2 shows that

$$2^{-(m+n)r} \leq L(p'q') \leq 1,$$

whence $p'q' \neq 0$, and $(pq)' = p'q'$.

If we now let $\lambda = 1/L(p'q')$, we have

$$L(\lambda p'q') = L(\lambda(pq)') = 1,$$

and so $\lambda pq \in P_{m+n}(r)$. Clearly λ lies in the required range.

Corollary 2:

Let $p \in P_n(r)$ and let k be a positive integer. Then there exists a complex number λ such that $\lambda p^k \in P_{nk}(r)$, and, for such λ , $1 \leq |\lambda| \leq 2^{nkr}$.

Proof:

As in the previous corollary, we form the polynomial p' . By the same reasoning as before,

$$\{p^k\}' = p'\{p^{k-1}\}' = \dots = \{p'\}^k.$$

Now, applying Theorem 3.2, with $p_j = p'$ ($j = 1, \dots, k$), we obtain, since $L(p') = 1$,

$$2^{-nkr} \leq L(\{p^k\}') \leq 1.$$

Putting $\lambda = 1/L(\{p^k\}')$, we obtain the result.

Definition:

Let A be a commutative unital Banach algebra over the field F ($= \mathbb{R}$ or \mathbb{C}). Let

$$\text{cap}_n(a_1, \dots, a_r) = \inf \{ \|p(a_1, \dots, a_r)\| : p \in P_n(r) \},$$

and define $\text{cap}(a_1, \dots, a_r)$, the joint capacity of (a_1, \dots, a_r) by

$$\text{cap}(a_1, \dots, a_r) = \lim_{n \rightarrow \infty} \inf \{ \text{cap}_n(a_1, \dots, a_r) \}^{\frac{1}{n}}.$$

It will often be convenient to denote the r -tuple (a_1, \dots, a_r) by \underline{a} and write $\text{cap } \underline{a}$ for $\text{cap}(a_1, \dots, a_r)$.

If $\text{cap } \underline{a} = 0$, we shall say the elements a_1, \dots, a_r of \underline{a} are jointly quasialgebraic; clearly the order of the elements in the r -tuple is immaterial.

Notice that we have defined the joint capacity in terms of the lower limit of $\{\text{cap}_n \underline{a}\}^{\frac{1}{n}}$, rather than the infimum, since it is the asymptotic behaviour of the quantity which is of interest. On the surface, there seems little evidence to suggest that $\{\text{cap}_n \underline{a}\}^{\frac{1}{n}}$ converges as $n \rightarrow \infty$, since all we know in general is that

$$\text{cap}_{m+n} \underline{a} \leq 2^{(m+n)r} \text{cap}_m \underline{a} \text{cap}_n \underline{a},$$

and that

$$\text{cap}_{m+n} \underline{a} \leq \min_{1 \leq i \leq r} \|a_i^m\| \text{cap}_n \underline{a},$$

two conditions which do not themselves force $\{\text{cap}_n \underline{a}\}^{\frac{1}{n}}$ to converge. (The first inequality follows since a product of two "monic" polynomials may only form a monic polynomial when multiplied by a factor of $2^{(m+n)r}$, in the usual notation for these integers, and the second on noting that if $p \in P_n(r)$, the $z_i^m p \in P_{m+n}(r)$.) On the other hand, we do not know of any example where the sequence $\{\text{cap}_n \underline{a}\}^{\frac{1}{n}}$ fails to converge.

A further question which may be raised at this stage is the size of the exponent $\frac{1}{n}$; in the case of a single variable, the number n represents both the degree of the polynomial and (asymptotically) the number of terms. In the several-variable case these quantities are not asymptotically equivalent, a polynomial of degree n having up to $\frac{1}{r!} (n+1) \dots (n+r)$ terms. Since $(n+1) \dots (n+r) \sim n^r$ as $n \rightarrow \infty$, there is reason to consider taking the $\frac{1}{r!} n^r$ -th root instead; to show that this does not lead to a satisfactory theory, we shall only be concerned whether or not the evaluated "capacity" is zero, so, for simplicity, we shall take the n^r -th root instead, and consider the quantity $\liminf_{n \rightarrow \infty} \{\text{cap}_n \underline{a}\}^{1/n^r}$.

Let p be a polynomial in r variables, of degree n ,

$$p(z_1, \dots, z_r) = \sum_{|k| \leq n} \alpha_k \underline{z}^k,$$

and write

$$p_\lambda(z_1, \dots, z_r) = \sum_{|k| \leq n} \alpha_k (\lambda^{n-|k|}) \underline{z}^k,$$

where $\lambda \in \mathbb{C} \setminus \{0\}$; p_λ is monic if and only if p is. Thus, $\forall \epsilon > 0$, $\exists p$ in $P_n(r)$ such that if $\lambda \neq 0$,

$$\begin{aligned} \text{cap}_n (\lambda a_1, \dots, \lambda a_r) + \epsilon &> \| p(\lambda a_1, \dots, \lambda a_r) \| \\ &= |\lambda|^n \| p_{\lambda^{-1}}(a_1, \dots, a_r) \| \\ &\geq |\lambda|^n \text{cap}_n (a_1, \dots, a_r) . \end{aligned}$$

But ϵ was arbitrary, so $\text{cap}_n (\lambda a_1, \dots, \lambda a_r) \geq |\lambda|^n \text{cap}_n(a_1, \dots, a_r)$, and since $\lambda^{-1}\lambda a_i = a_i$ ($i = 1, 2, \dots, r$), the reverse inequality is also true. But

$$\begin{aligned} \liminf \{ \text{cap}_n (\lambda a_1, \dots, \lambda a_r) \}^{1/n^r} &= \liminf |\lambda|^{1/n^{r-1}} \{ \text{cap}_n(a_1, \dots, a_r) \}^{1/n^r} \\ &= \liminf \{ \text{cap}_n (a_1, \dots, a_r) \}^{1/n^r} . \end{aligned}$$

It follows that $\liminf \{ \text{cap}_n (a_1, \dots, a_r) \}^{1/n^r}$ does not depend on $\| a_1 \|, \dots, \| a_r \|$, since we may choose λ arbitrarily large without altering the $\liminf \{ \text{cap}_n \}^{1/n^r}$. Further, this quantity bears no relation to the spectrum either, as we shall show in a moment; for these two reasons we discard it.

Counterexample:

To show the lack of relation between the spectrum and the quantity $\liminf \{ \text{cap}_n a \}^{1/n^r}$ we produce a pair (x, y) of elements of an algebra which have joint spectrum $(0, 0)$, but with

$$\liminf \{ \text{cap}_n a \}^{1/n^r} \geq 1 .$$

Let $\Delta = \{(z_1, z_2) \in \mathbb{C}^2 : |z_i| \leq 1, (i = 1, 2)\}$, and let A be the Banach algebra of all functions analytic on the interior of Δ and continuous on its boundary; we endow A with the uniform topology and the convolution product

$$(x \circ y)(\omega_1, \omega_2) = \int_0^{\omega_1} d\zeta_1 \int_0^{\omega_2} x(\omega_1 - \zeta_1, \omega_2 - \zeta_2) y(\zeta_1, \zeta_2) d\zeta_2,$$

where $(\omega_1, \omega_2) \in \Delta$ and the integration with respect to ζ_i is taken over any Jordan arc joining 0 to ω_i and (with the possible exception of ω_i) lying in the set $\{z_i : |z_i| < 1\}$ ($i = 1, 2$).

A is thus a radical algebra, for it is not hard to see that

$$\|x^n\| \leq \frac{\|x\|^n}{\{(n-1)!\}^2}.$$

Thus if $(x, y) \in \Delta$, the pair (x, y) has joint spectrum $(0, 0)$, and each of x, y have spectrum $\{0\}$. Let $x(z_1, z_2) = z_1$, and $y(z_1, z_2) = z_2 \forall (z_1, z_2) \in \Delta$. Then, for all positive integers m and n , except for the pair $(0, 0)$, we have

$$x^m \circ y^n(z_1, z_2) = \frac{z_1^{2m+n-1} z_2^{m+2n-1}}{(2m+n-1)! (m+2n-1)!}.$$

Let $p \in P_n(2)$, and write

$$p(\xi_1, \xi_2) = \sum_{|k| \leq n} \alpha_k \xi^k$$

where $\xi = (\xi_1, \xi_2)$, and let

$$p_0(z_1, z_2) = [p(x, y)](z_1, z_2),$$

for $(z_1, z_2) \in \Delta$, so p_0 considered as a function of z_1 and z_2 , is a polynomial of degree $3n-2$. Clearly

$$\|p(x, y)\| = \sup \{|p_0(z_1, z_2)| : (z_1, z_2) \in \Delta\}.$$

Now

$$p_0(z_1, z_2) = \sum_{|k| \leq 3n-2} \beta_k z^k,$$

where $\beta_k = \sum \{\alpha_{j_1, j_2} \cdot \frac{1}{(2j_1+j_2-1)! (j_1+2j_2-1)!} : 2j_1+j_2-1=k_1, j_1+2j_2-1=k_2\}$

Further each term of highest degree in p_0 arises from one term in p , that is, a term of degree $3n-2$ arises from the substitution for $x^m y^n$ in only one term of p . Now by Mahler's lemma (Lemma 3.1) applied to polynomials in two variables

$$\begin{aligned} |\beta_{k_1, k_2}| &\leq \binom{3n-2}{k_1} \binom{3n-2}{k_2} M_2(p_0) \\ &\leq \binom{3n-2}{k_1} \binom{3n-2}{k_2} \|p_0\|_{\Delta}, \end{aligned} \quad (1)$$

since $M_2(p_0) \leq \|p_0\|_\Delta$ by the maximum modulus theorem. But, if $n-1 \leq k \leq 2n-1$,

$$\beta_{k, 3n-2-k} = \frac{\alpha_{k-n+1, 2n-k-1}}{k!(3n-2-k)!},$$

and if k lies outside this range $\beta_{k, 3n-2-k} = 0$, on equating the coefficients of $z_1^k z_2^{3n-2-k}$, so we have, since p is monic,

$$1 = \sum_{k=0}^n |\alpha_{k, n-k}| = \sum_{k=0}^n |\beta_{n-1+k, 2n-k-1}| (n-1+k)! (2n-1-k)!.$$

Moreover, since $(n-1+k)! (2n-1-k)! \leq (n-1)! (2n-1)!$ for

$k = 0, 1, \dots, n$, we have

$$\begin{aligned} 1 &\leq (n-1)! (2n-1)! \sum_{k=0}^n |\beta_{n-1+k, 2n-k-1}| \\ &\leq (n-1)! (2n-1)! \sum_{k=0}^n \binom{3n-2}{n-1+k} \binom{3n-2}{2n-k-1} \|p_0\|_\Delta \quad \text{by (1)} \\ &\leq (n-1)! (2n-1)! (n+1) \max_k \binom{3n-2}{n-1+k}^2 \|p_0\|_\Delta \\ &= (n-1)! (2n-1)! (n+1) \binom{3n-2}{\frac{3n-1}{2}}^2 \|p_0\|_\Delta \quad (\text{if } n \text{ even}). \end{aligned}$$

Hence we can write

$$\|p_0\|_\Delta \geq \{(n-1)! (2n-1)! (n+1) [(3n-2)!]^2 \left(\frac{3n-1}{2}\right)!^{-4}\}^{-1} \quad (2).$$

But p was monic of degree n , and $\|p_0\|_\Delta = \|p(x, y)\|$, so $\text{cap}_n(x, y)$ exceeds the quantity on the right hand side of (2), and the $1/n^2$ -th power of this tends to 1 as $n \rightarrow \infty$, and noting the same conclusion if n is odd (when $\max_k \binom{3n-2}{k} = \binom{3n-2}{\frac{3n-1}{2}}$) we see, somewhat thankfully, that we have the desired conclusion that $\liminf \{\text{cap}_n(x, y)\}^{1/n^2} \geq 1$.

Having quoted this rather laborious example, we shall ignore the quantity obtained from the $1/n^r$ -th root, and proceed with the development of the theory of the joint capacity as originally defined.

§ 3. Spectral Capacity

In defining a spectral capacity we make the obvious extension of the definition of the Tchebycheff constant of a plane set to sets in \mathbb{C}^r analogous to our definition of the joint capacity of an r -tuple of Banach algebra elements.

Definition:

Let E be a compact subset of \mathbb{C}^r , and define

$$\text{Cap}_n E = \inf \{ \|p\|_E : p \in P_n(r) \},$$

and

$$\text{Cap } E = \lim_{n \rightarrow \infty} \inf \{ \text{Cap}_n E \}^{\frac{1}{n}}.$$

We shall call $\text{Cap } E$ the capacity of E .

Theorem 3.3

Let $\underline{a} = (a_1, \dots, a_r)$ be an r -tuple of elements of a complex commutative unital Banach algebra A ; then if $\text{Sp } \underline{a}$ denotes the joint spectrum of (a_1, \dots, a_r) we have

$$\text{cap } \underline{a} = 0 \text{ if and only if } \text{Cap } \text{Sp } \underline{a} = 0.$$

More specifically,

$$\text{Cap } \text{Sp } \underline{a} \leq \text{cap } \underline{a} \leq 2^r \text{Cap } \text{Sp } \underline{a}.$$

Proof:

We first observe that

$$\begin{aligned} \|p\|_{\text{Sp } \underline{a}} &= \sup \{ |p(\underline{\lambda})| : \underline{\lambda} \in \text{Sp } \underline{a} \} \\ &= \sup \{ |p(\phi(a_1), \dots, \phi(a_r))| : \phi \in \Phi_A \} \\ &= \sup \{ |\phi(p(a_1, \dots, a_r))| : \phi \in \Phi_A \} \\ &= \sup \{ |\zeta| : \zeta \in \text{Sp } p(a_1, \dots, a_r) \} \\ &= \rho(p(a_1, \dots, a_r)) \leq \|p(a_1, \dots, a_r)\|, \end{aligned}$$

where Φ_A denotes the carrier space of A . Hence, for any positive integer n ,

$$\begin{aligned} \text{Cap}_n \text{Sp } \underline{a} &= \inf \{ \| p \|_{\text{Sp } \underline{a}} : p \in P_n(r) \} \\ &\leq \inf \{ \| p(a_1, \dots, a_r) \| : p \in P_n(r) \} \\ &= \text{cap}_n \underline{a}, \end{aligned}$$

whence

$$\text{Cap Sp } \underline{a} \leq \text{cap } \underline{a}.$$

Now let $\epsilon > 0$; for each positive integer k , there exists a polynomial s_k belonging to $P_k(r)$ such that

$$\| s_k \|_{\text{Sp } \underline{a}} < \{ (\text{Cap}_k \text{Sp } \underline{a})^{\frac{1}{k}} + \epsilon \}^k.$$

Two cases arise; when $\| s_k \|_{\text{Sp } \underline{a}} = 0$ for some integer k , and where $\| s_k \|_{\text{Sp } \underline{a}} \neq 0$ for all positive integers k .

We consider the latter case first: Since $\| s_k \|_{\text{Sp } \underline{a}} = \rho(s_k(\underline{a})) \neq 0$ there is an integer $N = N(k)$ for which

$$\| \{ s_k(\underline{a}) \}^n \|_{\frac{1}{nk}} < 2\rho(s_k(\underline{a})) = 2\| s_k \|_{\text{Sp } \underline{a}} \quad \forall n \geq N, \quad (1)$$

where $s_k(\underline{a})$ denotes $s_k(a_1, \dots, a_r)$, an abbreviation we will use henceforth.

Now write t_{nk} for a constant multiple of $\{ s_k \}^n$ which is monic, so $t_{nk} = \lambda_n s_k^n$ for some λ_n with $1 \leq |\lambda_n| \leq 2^{nkr}$, by the second Corollary to Theorem 3.2. Therefore, if $n \geq N(k)$,

$$\begin{aligned} \| t_{nk}(\underline{a}) \|_{\frac{1}{nk}} &\leq 2^r \| \{ s_k(\underline{a}) \}^n \|_{\frac{1}{nk}} \\ &< 2^{r+\frac{1}{k}} \| s_k \|_{\text{Sp } \underline{a}}^{\frac{1}{k}} \quad \text{by (1)} \\ &< 2^{r+\frac{1}{k}} \{ (\text{Cap}_k \text{Sp } \underline{a})^{\frac{1}{k}} + \epsilon \}. \end{aligned}$$

Therefore, since $t_{nk} \in P_{nk}(r)$,

$$\{ \text{cap}_{nk} \underline{a} \}^{\frac{1}{nk}} \leq 2^{r+\frac{1}{k}} \{ (\text{Cap}_k \text{Sp } \underline{a})^{\frac{1}{k}} + \epsilon \}. \quad (2)$$

Now, since n is a positive integer, $nk \geq k$, so $nk \rightarrow \infty$ as

$k \rightarrow \infty$ no matter what choice is made of $n \geq N(k)$. Thus, letting $k \rightarrow \infty$ in (2), we obtain

$$\text{cap } \underline{a} \leq \liminf_{k \rightarrow \infty} (\text{cap}_{nk} \underline{a})^{\frac{1}{nk}} \leq 2^r (\text{Cap Sp } \underline{a} + \epsilon).$$

But ϵ was arbitrary, so

$$\text{cap } \underline{a} \leq 2^r \text{Cap Sp } \underline{a}.$$

In the other case, we have $\|s_k\|_{\text{Sp } \underline{a}} = 0$, so let λ_j be a complex number such that $\lambda_j \{s_k\}^j \in P_{jk}(r)$; we know that $1 \leq \lambda_j \leq 2^{jkr}$. Then $\text{cap}_{jk} \underline{a} \leq \|\lambda_j s_k^j(\underline{a})\| \leq 2^{jkr} \|s_k^j(\underline{a})\|$, and we have

$$(\text{cap}_{jk} \underline{a})^{\frac{1}{jk}} \leq 2^r (\|s_k^j(\underline{a})\|)^{\frac{1}{j}}$$

Hence

$$\begin{aligned} \liminf_{j \rightarrow \infty} (\text{cap}_{jk} \underline{a})^{\frac{1}{jk}} &\leq 2^r \{\rho(s_k(\underline{a}))\}^{\frac{1}{k}} \\ &= 2^r \{\|s_k\|_{\text{Sp } \underline{a}}\}^{\frac{1}{k}} \\ &= 0, \end{aligned}$$

and we have $\text{cap } \underline{a} = 0$, so in particular $\text{cap } \underline{a} \leq 2^r \text{Cap Sp } \underline{a}$, and the theorem is proved.

Theorem 3.4

If there is a non-zero polynomial p for which $p(a_1, \dots, a_r)$ is quasialgebraic (in the original sense), then the r -tuple (a_1, \dots, a_r) is jointly quasialgebraic.

Proof:

We may assume, by multiplying by a constant if required, that p is monic, say $p \in P_k(r)$. Now let $q \in P_n(1)$, a polynomial in a single variable, so that there is a constant λ with $\lambda(q \circ p) \in P_{nk}(r)$, for let $p = p' + p_0$, where p' contains all the terms of p of degree k , and p_0 contains all the terms of degree $k-1$ or less; clearly all the terms of $q \circ p$ of degree nk are in the polynomial $(p')^n$,

and $(p')^n$ consists entirely of terms of degree nk . Using Mahler's notation as in Theorem 3.2, it will follow that $\lambda(q \circ p)$ will be monic if $\lambda^{-1} = L(\{p'\}^n)$. But, by Theorem 3.2, Corollary 2, we have

$$L(\{p'\}^n) \leq \{L(p')\}^n \leq 2^{nkr} L(\{p'\}^n),$$

so, since $L(p') = 1$ because $p \in P_k(r)$, $1 \leq |\lambda| \leq 2^{nkr}$. From this we see that

$$\begin{aligned} \text{cap}_{nk} \underline{a} &= \inf \{ \|s(\underline{a})\| : s \in P_{nk}(r) \} \\ &\leq \| \lambda(q \circ p)(\underline{a}) \| \\ &\leq 2^{nkr} \| q(p(\underline{a})) \| . \end{aligned}$$

But q was any member of $P_n(1)$, so

$$\begin{aligned} \text{cap}_{nk} \underline{a} &\leq 2^{nkr} \inf \{ \| q(p(\underline{a})) \| : q \in P_n(1) \} \\ &= 2^{nkr} \text{cap}_n p(\underline{a}), \end{aligned}$$

and

$$\begin{aligned} \text{cap} \underline{a} &\leq \liminf_{n \rightarrow \infty} (\text{cap}_{nk} \underline{a})^{\frac{1}{nk}} \\ &\leq 2^r \liminf_{n \rightarrow \infty} (\text{cap}_n p(\underline{a}))^{\frac{1}{nk}} \\ &= 0, \end{aligned}$$

since $\text{cap} p(\underline{a}) = 0$.

At this stage we may wonder whether the converse of this theorem is true, which is the case with polynomials of one variable (Theorem 2.6). Since this would degenerate the property of being jointly quasialgebraic to one of the existence of a non-zero polynomial p in r variables such that $p(a_1, \dots, a_r)$ is quasialgebraic, it is important to show that this is not so. We require a lemma.

Lemma 3.5

Let E be an arcwise-connected compact subset of \mathbb{C} ; then the algebra $C(E)$ has the following properties:-

- (i) If $f \in C(E)$, $Sp f = f(E)$,
 (ii) $cap f = 0$ if and only if $f(E)$ contains only one point.

Proof:

(i) is obvious.

(ii) Clearly, if $f(E) = Sp f = \{\alpha\}$, then $cap f = 0$; we have to show that if $Sp f$ contains two distinct points then $cap f > 0$, or by Theorem 2.5, $Cap Sp f > 0$. Let $z_1, z_2 \in E$ be points such that $f(z_1) \neq f(z_2)$; by multiplying f by a scalar of modulus 1 if required, we may assume that $Re f(z_1) \neq Re f(z_2)$. Now E is arcwise connected, so there exists a continuous function $\phi: [0, 1] \rightarrow E$ such that $\phi(0) = z_1$, $\phi(1) = z_2$. Let $F: [0, 1] \rightarrow \underline{\mathbb{R}}$ be defined by

$$F(t) = Re f(\phi(t)) .$$

F is continuous, and by the intermediate value theorem, $F([0, 1])$ contains the interval $[Re f(z_1), Re f(z_2)]$ (or $[Re f(z_2), Re f(z_1)]$). Now let

$$\Sigma = \{ z \in Sp f : Re z \in F([0, 1]) \} .$$

Since $\Sigma \subseteq Sp f$, $Cap \Sigma \leq Cap Sp f$, and using the transfinite diameter we have

$$\begin{aligned} \delta_n^{\frac{1}{2^{n(n-1)}}}(\Sigma) &= \sup \left\{ \prod_{1 \leq i < j \leq n} |z_i - z_j| : z_i \in \Sigma \right\} \\ &\geq \sup \left\{ \prod_{1 \leq i < j \leq n} |Re z_i - Re z_j| : z_i \in \Sigma \right\} \\ &= \delta_n^{\frac{1}{2^{n(n-1)}}}(F([0, 1])) . \end{aligned}$$

Therefore,

$$\begin{aligned} Cap \Sigma &= \lim_n \delta_n(\Sigma) \geq \lim_n \delta_n(F([0, 1])) \\ &= Cap F([0, 1]) \\ &> 0, \end{aligned}$$

since $F([0, 1])$ contains an interval. Thus $cap f > 0$.

Counterexample:

There exists a Banach algebra A and a pair (f, g) of elements of A with the properties

(i) $\text{cap}(f, g) = 0$,

(ii) there is no polynomial other than the zero polynomial for which $p(f, g)$ is quasia algebraic.

Let $\Delta = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$, $D = \{z \in \mathbb{C} : |z| < 1\}$ and A be the algebra $C(\Delta)$. Define f, g as follows:

$$f(z) = \sum_{k=0}^{\infty} z^{m(k)} \quad \forall z \in D,$$

$$g(z) = z \quad \forall z \in D,$$

where $m(k) = 2^{2^k}$. Both f and g are analytic in D and, in particular, their restrictions to Δ , which we shall denote also by f and g , are continuous, so $f, g \in A$.

First, we show that f and g are jointly quasia algebraic. Let

$$p_n(z_1, z_2) = z_1 - \sum_{k=0}^n z_2^{m(k)},$$

so that $p_n \in P_{m(n)}(2)$, since the only term of highest degree is $z_2^{m(k)}$. Let $F_n(z) = p_n(f(z), g(z))$, so that

$$\begin{aligned} F_n(z) &= f(z) - \sum_{k=0}^n z^{m(k)} \\ &= \sum_{k=n+1}^{\infty} z^{m(k)}. \end{aligned}$$

Now,

$$\begin{aligned} \|p_n(f, g)\| &= \sup \{|F_n(z)| : z \in \Delta\} \\ &\leq \sum_{k=n+1}^{\infty} \left(\frac{1}{2}\right)^{m(k)} \\ &< \sum_{k=m(n+1)}^{\infty} \left(\frac{1}{2}\right)^k \\ &= 2 \cdot \left(\frac{1}{2}\right)^{m(n+1)}, \end{aligned}$$

whence it follows that

$$\text{cap}_{m(n)}(f, g) \leq \|p_n(f, g)\| \leq 2 \cdot \left(\frac{1}{2}\right)^{m(n+1)},$$

and

$$\begin{aligned} \text{cap}(f, g) &= \liminf_{k \rightarrow \infty} \{\text{cap}_k(f, g)\}^{\frac{1}{k}} \\ &\leq \liminf_{n \rightarrow \infty} \{\text{cap}_{m(n)}(f, g)\}^{\frac{1}{m(n)}} \\ &\leq \liminf_{n \rightarrow \infty} \left\{ 2^{\frac{1}{m(n)}} \cdot \left(\frac{1}{2}\right)^{\frac{m(n+1)}{m(n)}} \right\} \\ &\leq \liminf_{n \rightarrow \infty} \left\{ 2^{\frac{1}{m(n)}} \left(\frac{1}{2}\right)^{m(n)} \right\} \\ &= 0, \end{aligned}$$

$$\text{since } m(n+1) = 2^{2^{n+1}} = \left(2^{2^n}\right)^2 = \{m(n)\}^2.$$

Before we deal with the question whether there is a non-zero polynomial p for which $p(f, g)$ is quasia algebraic, we discuss some properties of the function f .

Let ω be a complex number of modulus 1 such that $\omega^{2^r} = 1$ for some integer r . Then $\omega^{m(k)} = 1$ for $k \geq \log_2 r$. Let

$$\Omega = \{z \in \mathbb{C} : z^{2^r} = 1 \text{ for some positive integer } r\},$$

and let $\omega \in \Omega$, $t \in (0, 1)$; we have

$$\begin{aligned} f(t\omega) &= \sum_{k=0}^{\infty} (t\omega)^{m(k)} \\ &= (t\omega)^2 + (t\omega)^4 + \dots + (t\omega)^{m(r-1)} + t^{m(r)} + \dots \\ &= h_1(t) + h_2(t), \end{aligned}$$

where $h_1(t) = \sum_{k=1}^{r-1} (t\omega)^{m(k)}$, $h_2(t) = \sum_{k=r}^{\infty} t^{m(k)}$ and $\omega^{m(r)} = 1$.

Now h_1 , being a polynomial in t , is bounded on the set $[0, 1]$,

and $h_2(t) \rightarrow \infty$ as $t \rightarrow 1^-$, since

$$\begin{aligned} \lim_{t \rightarrow 1^-} h_2(t) &= \sup_{0 < t < 1} h_2(t) = \sup_{0 < t < 1} \sum_{k=r}^{\infty} t^{m(k)} \\ &= \sup_{0 < t < 1} \sup_{N > r} \sum_{k=r}^N t^{m(k)} \end{aligned}$$

and we may interchange the order of the suprema. Therefore

$|f(t\omega)| \rightarrow \infty$ as $t \rightarrow 1^-$, for all $\omega \in \Omega$; we observe that Ω is an infinite subset of the unit circle.

Now assume the existence of a non-zero polynomial p such that $p(f, g)$ is quasialgebraic. Let $F(z) = p(f(z), g(z))$, so that $F \in C(\Delta)$ and $\text{cap } F = 0$, so by Lemma 3.5, $F(\Delta)$ contains only one point; by subtracting a constant if necessary, we may assume that $F(z) = 0$ for all $z \in \Delta$. But $F = p(f, g)$, so F is the restriction to Δ of a function analytic in D , hence $F(z) = 0$ for all $z \in D$, where we do not distinguish between the function defined on D and that defined on Δ . Write

$$p(z_1, z_2) = z_1^n a_n(z_2) + z_1^{n-1} a_{n-1}(z_2) + \dots + a_0(z_2),$$

where a_0, \dots, a_n are polynomials in z_2 , and $a_n \not\equiv 0$. Now $g(z) = z$, so $n \geq 1$, for otherwise $[p(f, g)](z) = a_0(z)$ and this cannot be identically zero in D if p is non-zero.

Now $F(z) = 0 \quad \forall z \in D$; let $\omega \in \Omega$, $t \in (0, 1)$ so $t\omega \in D$. Since $|f(t\omega)| \rightarrow \infty$ as $t \rightarrow 1^-$, $|f(t\omega)| > 1$ for $t_0 < t < 1$, so,

$$\begin{aligned} 0 = F(t\omega) &= p(f(t\omega), g(t\omega)) \\ &= a_n(t\omega) \{f(t\omega)\}^n + \dots + a_0(t\omega) \quad (t_0 < t < 1). \end{aligned}$$

Hence

$$a_n(t\omega) + \frac{1}{f(t\omega)} a_{n-1}(t\omega) + \dots + \frac{1}{\{f(t\omega)\}^n} a_0(t\omega) = 0 \quad (t_0 < t < 1)$$

and, keeping ω fixed and letting $t \rightarrow 1^-$, we have, since

$a_k(t\omega) \rightarrow a_k(\omega)$ ($k = 0, 1, \dots, n$) as $t \rightarrow 1^-$, (a_k being a polynomial),

$$\lim_{t \rightarrow 1^-} a_n(t\omega) = a_n(\omega) = 0.$$

Therefore, $a_n(\omega) = 0 \quad \forall \omega \in \Omega$; but a_n is a polynomial and so a_n is identically zero, since Ω is an infinite set. This contradicts

the definition of a_n , which was not identically zero; therefore there is no non-zero polynomial p for which $p(f, g)$ is quasialgebraic, and the counterexample is complete.

§ 4. A perturbation theorem.

Let Λ be a commutative unital complex Banach algebra, and let Φ_Λ be its carrier space with the usual (weak*) topology. We define the derived spectrum, $\sigma(a)$, of an element $a \in \Lambda$ by

$$\sigma(a) = \{\phi(a) : \phi \in \Phi'_\Lambda\},$$

where Φ'_Λ is the derived set of Φ_Λ . Clearly $\sigma(a)$ is a compact subset of $\text{Sp } a$. For an r -tuple (a_1, \dots, a_r) we also define the derived joint spectrum

$$\sigma(a_1, \dots, a_r) = \{(\phi(a_1), \dots, \phi(a_r)) : \phi \in \Phi'_\Lambda\}.$$

Let $J = \{a \in \Lambda : \sigma(a) = \{0\}\}$. J is a closed ideal of Λ , so let $\pi: \Lambda \rightarrow \Lambda/J$ be the natural epimorphism.

Lemma 3.6

With the above notation,

$$\text{Sp } (\pi(a_1), \dots, \pi(a_r)) = \sigma(a_1, \dots, a_r).$$

Proof:

We need only show that $0 \in \text{Sp } (\pi(a_1), \dots, \pi(a_r)) \iff 0 \in \sigma(a_1, \dots, a_r)$.

If $0 \notin \text{Sp } (\pi(a_1), \dots, \pi(a_r))$, then

$$(\Lambda/J)\pi(a_1) + \dots + (\Lambda/J)\pi(a_r) = \Lambda/J,$$

and so there exist $b_1, \dots, b_r \in \Lambda$ such that

$$\pi(b_1)\pi(a_1) + \dots + \pi(b_r)\pi(a_r) = \pi(1).$$

Hence, we have successively

$$b_1 a_1 + \dots + b_r a_r - 1 \in J,$$

$$\begin{aligned}\phi(b_1)\phi(a_1) + \dots + \phi(b_r)\phi(a_r) - 1 &= 0 \quad \forall \phi \in \Phi'_\Lambda, \\ (\phi(a_1), \dots, \phi(a_r)) &\neq (0, \dots, 0) \quad \forall \phi \in \Phi'_\Lambda, \\ 0 &\notin \sigma(a_1, \dots, a_r) .\end{aligned}$$

Conversely, let $0 \notin \sigma(a_1, \dots, a_r)$. Then

$$\Psi = \{ \phi : (\phi(a_1), \dots, \phi(a_r)) = (0, \dots, 0) \} \subseteq \Phi_\Lambda \setminus \Phi'_\Lambda .$$

Ψ is clearly a compact set, and since $\Psi \subseteq \Phi_\Lambda \setminus \Phi'_\Lambda$, it is open, so the Silov idempotent theorem shows the existence of an idempotent e in Λ such that $\phi(e) = 1$ iff $\phi \in \Psi$. Then

$$(\phi(a_1 + e), \dots, \phi(a_r + e)) = \begin{cases} (\phi(a_1), \dots, \phi(a_r)) & (\text{if } \phi \notin \Psi) \\ (1, \dots, 1) & (\text{if } \phi \in \Psi) , \end{cases}$$

so $(0, \dots, 0) \notin \text{Sp}(a_1 + e, \dots, a_r + e)$. So there exist $b_1, \dots, b_r \in \Lambda$ such that

$$b_1(a_1 + e) + \dots + b_r(a_r + e) = 1 ,$$

and hence, since $\pi(e) = 0$,

$$\pi(b_1)\pi(a_1) + \dots + \pi(b_r)\pi(a_r) = \pi(1) ,$$

whence $0 \notin \text{Sp}(\pi(a_1), \dots, \pi(a_r))$.

Lemma 3.7

Let Λ and B be two compact subsets of \mathbb{C}^r , with $\Lambda \subseteq B$ and $B \setminus \Lambda$ containing only a finite number of points. Then

$$\text{Cap } \Lambda = \text{Cap } B .$$

Proof:

We need only show that the result holds if $B \setminus \Lambda$ consists of a single point; $B \setminus \Lambda = \{(\lambda_1, \dots, \lambda_r)\}$. We already know that $\text{Cap } \Lambda \leq \text{Cap } B$, so we must prove the opposite inequality.

Let $\epsilon > 0$. Then there is a polynomial p in $P_n(r)$ such that

$$\|p\|_\Lambda < \{(\text{Cap } \Lambda)^{\frac{1}{n}} + \epsilon\}^n .$$

Let

$$q(z_1, \dots, z_r) = (z_1 - \lambda_1)p(z_1, \dots, z_r) ;$$

then q belongs to $P_{n+1}(r)$ and $q(\lambda_1, \dots, \lambda_r) = 0$,

$$\begin{aligned} \|q\|_B &= \|q\|_A = \sup_{z \in \Lambda} |z_1 - \lambda_1| \cdot |p(z_1, \dots, z_r)| \\ &\leq d \|p\|_A , \end{aligned}$$

where $d = \sup_{z \in \Lambda} |z_1 - \lambda_1|$, a constant. Therefore

$$\text{cap}_{n+1} B \leq \|q\|_B \leq d \|p\|_A \leq d \{(\text{cap}_n \Lambda)^{\frac{1}{n}} + \epsilon\}^n .$$

But this holds for any positive integer n , so

$$(\text{cap}_{n+1} B)^{\frac{1}{n+1}} \leq d^{\frac{1}{n+1}} \{(\text{cap}_n \Lambda)^{\frac{1}{n}} + \epsilon\}^{\frac{n}{n+1}} ,$$

and

$$\begin{aligned} \liminf_{n \rightarrow \infty} (\text{cap}_{n+1} B)^{\frac{1}{n+1}} &\leq \liminf_{n \rightarrow \infty} d^{\frac{1}{n+1}} \{(\text{cap}_n \Lambda)^{\frac{1}{n}} + \epsilon\}^{\frac{n}{n+1}} \\ &= \text{Cap } \Lambda + \epsilon , \end{aligned}$$

if $d > 0$; if $d = 0$, either $\Lambda = B = \{\lambda_1, \dots, \lambda_r\}$, or we may choose $d' = \sup |z_i - \lambda_i|$ for some suitable i .

But ϵ was arbitrary, so $\text{Cap } B \leq \text{Cap } \Lambda$, and we are finished.

Theorem 3.8

Let Λ be a complex commutative unital Banach algebra, and write $J = \{a \in \Lambda : \sigma(a) \subseteq \{0\}\}$. Then if $\pi: \Lambda \rightarrow \Lambda/J$ is the canonical epimorphism,

$$\text{cap}(a_1, \dots, a_r) = 0 \text{ if and only if } \text{cap}(\pi(a_1), \dots, \pi(a_r)) = 0 .$$

Proof:

We first dispose of the case where $\sigma(a) = \emptyset$, that is $\Phi'_\Lambda = \emptyset$. In this case, Φ_Λ , being a compact set, must be finite, and hence $\text{Sp } a$ is finite for all $a \in \Lambda$, so that every element of Λ is quasi-algebraic and the result is trivial. We shall assume this is not the

case.

By Lemma 3.6 and Theorem 3.3,

$$\text{cap}(a_1, \dots, a_r) = 0 \iff \text{Cap Sp}(a_1, \dots, a_r) = 0,$$

$$\text{cap}(\pi(a_1), \dots, \pi(a_r)) = 0 \iff \text{Cap } \sigma(a_1, \dots, a_r) = 0,$$

so it is enough to show that $\text{Cap Sp}(a_1, \dots, a_r) = 0$ if and only if $\text{Cap } \sigma(a_1, \dots, a_r) = 0$. Since $\sigma(a_1, \dots, a_r) \subseteq \text{Sp}(a_1, \dots, a_r)$, we need only prove that $\text{Cap Sp}(a_1, \dots, a_r) = 0$ whenever $\text{Cap } \sigma(a_1, \dots, a_r) = 0$.

Let $\text{Cap } \sigma(a_1, \dots, a_r) = 0$. Then, for all $\epsilon > 0$, there exists a positive integer n such that $\{\text{Cap}_n \sigma(\underline{a})\}^{\frac{1}{n}} < \epsilon$, and hence there is a polynomial p in $P_n(r)$ with $\|p\|_{\sigma(\underline{a})} < \epsilon^n$. Now

$$\begin{aligned} \|p\|_{\sigma(\underline{a})} &= \sup \{ |p(\underline{\lambda})| : \underline{\lambda} \in \sigma(\underline{a}) \} \\ &= \sup \{ |p(\phi(a_1), \dots, \phi(a_r))| : \phi \in \Phi'_\Lambda \} \\ &= \sup \{ |\phi(p(\underline{a}))| : \phi \in \Phi'_\Lambda \}. \end{aligned}$$

Therefore if we let $\Psi = \{\phi \in \Phi'_\Lambda : |\phi(p(\underline{a}))| < \epsilon^n\}$, then $\Psi \supseteq \Phi'_\Lambda$; Ψ is open, and since it contains Φ'_Λ it is also closed. Let

$$\omega(\underline{a}) = \{(\phi(a_1), \dots, \phi(a_r)) : \phi \in \Psi\},$$

so that $\sigma(\underline{a}) \subseteq \omega(\underline{a}) \subseteq \text{Sp } \underline{a}$, and

$$\|p\|_{\omega(\underline{a})} = \sup \{ |\phi(p(\underline{a}))| : \phi \in \Psi \} \leq \epsilon^n.$$

Now p was in $P_n(r)$, so there exists a constant θ_k with $1 \leq \theta_k \leq 2^{nkr}$ such that $\theta_k p^k$ belongs to $P_{nk}(r)$, so

$$\begin{aligned} \text{Cap}_{nk} \omega(\underline{a}) &\leq \|\theta_k p^k\|_{\omega(\underline{a})} \\ &\leq 2^{nkr} \epsilon^{nk}. \end{aligned}$$

Thus we have

$$\begin{aligned} \text{Cap } \omega(\underline{a}) &\leq \liminf_{k \rightarrow \infty} \{\text{Cap}_{nk} \omega(\underline{a})\}^{\frac{1}{nk}} \\ &\leq 2^r \epsilon. \end{aligned}$$

Now

$$\text{Sp } \underline{a} = \{(\phi(a_1), \dots, \phi(a_r)) : \phi \text{ in } \Phi_{\Lambda}\},$$

$$\omega(\underline{a}) = \{(\phi(a_1), \dots, (a_r)) : \phi \text{ in } \Psi\},$$

so

$$\text{Sp } \underline{a} \setminus \omega(\underline{a}) \subseteq \{(\phi(a_1), \dots, \phi(a_r)) : \phi \text{ in } \Phi_{\Lambda} \setminus \Psi\}.$$

But $\Phi_{\Lambda} \setminus \Psi$ is a finite set, because it is a compact set containing no limit points (since it does not meet Φ'_{Λ}). Hence, by Lemma 3.7, $\text{Cap Sp } \underline{a} = \text{Cap } \omega(\underline{a})$, and $\text{Cap Sp } \underline{a} \leq 2^r \epsilon$. But ϵ was arbitrary, so $\text{Cap Sp } \underline{a} = 0$, and the theorem is proved.

Corollary:

Let b_1, \dots, b_r belong to J . Then for any elements a_1, \dots, a_r of Λ , we have

$$\text{cap}(a_1, \dots, a_r) = 0 \text{ if and only if } \text{cap}(a_1 + b_1, \dots, a_r + b_r) = 0.$$

To relate this result to the original idea of capacity and the results of Chapter 2, we consider the idea of a Riesz actor of a Banach algebra. Let Λ be a Banach algebra, with $a \in \Lambda$, and define $\tilde{a}: \Lambda \rightarrow \Lambda$ by $\tilde{a}(x) = ax \forall x \in \Lambda$. This is a bounded linear operator on the Banach space Λ . If \tilde{a} is a Riesz operator, we shall call a a Riesz actor. We now have:

Theorem 3.9 (Smyth [15])

Let Λ be a complex commutative unital Banach algebra. Then if $a \in \Lambda$ is a Riesz actor, $\sigma(a) \subseteq \{0\}$. If, in addition, Λ is semi-simple, the converse holds.

This theorem, together with Theorem 3.8 means that, in a complex commutative unital Banach algebra, perturbation by a Riesz ^{actor} leaves

the joint capacity essentially invariant, (in the sense of being zero or non-zero), an analogue of Theorem 2.11.

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